ALGEBRAIC K-THEORY

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1. The Q construction

1.1. Introduction

References: (1) Weibel, C. *Intro to Algebraic K-theory*, in-progress, but available online; (2) Quillen, D. "Higher algebraic K-theory: I" in Springer LNM v.341, 1973; (3) Grason, D. "Higher algebraic K-theory: II" in Springer LNM v.551, 1976; (4) Srinivas, V. *Algebraic K-Theory*, Second Edition, Birkhauser, 1996.

We have seen a definition of the higher K-groups of a ring as the homotopy groups of a space:

$$K_i(R) = \pi_i(BGl(R)^+ \times K_0(R)).$$

Quillen was able to calculate the K-theory of finite fields with this definition, and Borel calculated the ranks of the rational K-groups of the ring of integers in a number field (=finite extension of $\mathbb{Q}$).

On the other hand, one would like to extend some of the fundamental structure theorems from classical K-theory to the higher K-groups, and so we need a more general construction.

1.2. Exact Categories

Definition 1. An *exact category* is an additive category $\mathscr{C}$ with a family $\mathcal{E}$ of sequences

$$0 \to B \xrightarrow{i} C \xrightarrow{j} D \to 0$$

such that there is an embedding of $\mathscr{C}$ as a full subcategory of some abelian category $\mathcal{A}$ satisfying

1. A sequence of the above form in $\mathscr{C}$ is in $\mathcal{E}$ if and only if the sequence is a short exact sequence in $\mathcal{A}$.
2. $\mathscr{C}$ is closed under extensions in $\mathcal{A}$ (i.e. if we have a short exact sequence $0 \to B \to C \to D \to 0$ in $\mathcal{A}$ with $B, D \in \mathscr{C}$ then $C \in \mathscr{C}$, up to isomorphism).

Example 1. (a) Any abelian category is of course exact.

(b) Any additive category can be made into an exact category by taking the exact sequences to be the split exact sequences.

(c) For any ring R, the category $\mathcal{P}(R)$ of finitely generated left projective modules over R is exact (the exact sequences are the split exact ones).

(d) For any (left) Noetherian ring R, the category $\mathcal{M}(R)$ of finitely generated left R-modules is abelian and therefore exact.

If $\mathscr{C}$ is an exact category, we will call a monomorphism (epimorphism) appearing in a short exact sequence an *admissible monomorphism* (*epimorphism*), and we will write $B \rightarrowtail C$ ($C \twoheadrightarrow D$). We will need the following

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Lemma 1. *Admissible epi's are closed under composition and base change; in addition, they are closed under cobase change along an admissible mono. There is a corresponding dual statement for admissible mono's.*

Definition 2. Given an exact category $\mathscr{C}$, we define the Grothedieck group $K_0(\mathscr{C})$ to be $\mathcal{F}/\mathcal{R}$, where $\mathcal{F}$ is the free abelian group on isomorphism classes of $\mathscr{C}$ and $\mathcal{R}$ is the subgroup generated by $[C]-[B]-[D]$ for each short exact sequence

$$0 \to B \to C \to D \to 0$$

in $\mathscr{C}$.

Of course, when $\mathscr{C} = \mathcal{P}(R)$ for some ring R, this agrees with the old definition.

1.3. The Q construction

In order to define the higher K-groups of an exact category, we will first define an auxiliary category $Q\mathscr{C}$.

Definition 3. Given an exact category $\mathscr{C}$, we define a new category $Q\mathscr{C}$ by taking the objects of $Q\mathscr{C}$ to be the objects of $\mathscr{C}$. Given $A, B \in Q\mathscr{C}$ a morphism from A to B in $Q\mathscr{C}$ will be an isomorphism class of diagrams

$$A \overset{q}{\twoheadleftarrow} C \overset{i}{\rightarrowtail} B$$

in $\mathscr{C}$. Now given such a morphism $A \to B$ and a morphism $B \to D$

$$B \overset{p}{\twoheadleftarrow} E \overset{j}{\rightarrowtail} D$$

their composite is defined by

$$\begin{tikzcd} & C \times_B E \arrow[r, tail] \arrow[d, two heads] & E \arrow[r, tail, "j"] \arrow[d, two heads, "p"] & D \\ A & C \arrow[l, two heads, "q"] \arrow[r, tail, "i"] & B & \end{tikzcd}$$

Note that we have used the lemma to know that the pullback of p and i are of the right type.

Remark 1. We think of a morphism $A \to B$ as an identification of A with a subquotient of B.

Remark 2. Any admissiblie mono $i : A \rightarrowtail B$ in $\mathscr{C}$ gives a morphism $A \to B$ in $Q\mathscr{C}$, namely $A \overset{\mathrm{id}}{\twoheadleftarrow} A \overset{i}{\rightarrowtail} B$. Similarly, any $q : B \twoheadrightarrow A$ in $\mathscr{C}$ gives a wrong way morphism $A \to B$ in $Q\mathscr{C}$, namely $A \overset{q}{\twoheadleftarrow} B \overset{\mathrm{id}}{\rightarrowtail} B$. As a result, any morphism $A \twoheadleftarrow C \rightarrowtail B$ in $Q\mathscr{C}$ can be regarded as a composite of two morphisms *in* $Q\mathscr{C}$.

Theorem 1. *For an exact category $\mathscr{C}$, we have $\pi_1(BQ\mathscr{C}) \cong K_0(\mathscr{C})$.*

Proof. We take $0 \in Q\mathscr{C}$ as our basepoint. We can consider any morphism $A \to B$ in $Q\mathscr{C}$ as an element of $\pi_1(BQ\mathscr{C})$ by concatenating this path with $0 \rightarrowtail A$ and the inverse of $0 \rightarrowtail B$. Then $\pi_1(BQ\mathscr{C})$ is generated by morphisms in $Q\mathscr{C}$ subject to the relation $[g] \cdot [f] = [g \circ f]$.

Consider a morphism $A \to B$ in $Q\mathscr{C}$, given by morphisms $A \twoheadleftarrow C \rightarrowtail B$ in $\mathscr{C}$. Then

$$[0 \rightarrowtail B]^{-1}[C \rightarrowtail B][A \twoheadleftarrow C][0 \rightarrowtail A] = [0 \rightarrowtail C]^{-1}[A \twoheadleftarrow C][0 \rightarrowtail A].$$

The composition $0 \rightarrowtail A \twoheadleftarrow C$ in $Q\mathscr{C}$ is given by $0 \twoheadleftarrow K \rightarrowtail C$, where $K \rightarrowtail C \twoheadrightarrow A$ is exact. Thus the above loop can alternatively be given by

$$[0 \rightarrowtail C]^{-1}[K \rightarrowtail C][0 \twoheadleftarrow K] = [0 \rightarrowtail K]^{-1}[0 \twoheadleftarrow K].$$

It follows that $\pi_1(BQ\mathscr{C})$ is generated by loops of the form $[0 \rightarrowtail A]^{-1}[0 \twoheadleftarrow A]$. We will denote these by L_A.

We claim that the relation $[g] \cdot [f] = [g \circ f]$ for all $f, g \in Q\mathscr{C}$ is equivalent to $L_A L_C = L_B$ for every $A \rightarrowtail B \twoheadrightarrow C$. Suppose we have $[g] \cdot [f] = [g \circ f]$ for all f, g and let $A \rightarrowtail B \twoheadrightarrow C$. Take f to be $0 \twoheadleftarrow C \overset{\mathrm{id}}{\rightarrowtail} C$ and g to be $C \twoheadleftarrow B$. Then $g \circ f$ is given by $0 \twoheadleftarrow C \twoheadleftarrow B \rightarrowtail B$. As we have seen,

$$[0 \rightarrowtail C]^{-1}[f] = L_C, \qquad [0 \rightarrowtail B]^{-1}[g][0 \rightarrowtail C] = L_A, \qquad [0 \rightarrowtail B]^{-1}[g \circ f] = L_B,$$

so $L_B = L_A L_C$.

On the other hand, suppose given this latter relation for every exact sequence and let $f = A \twoheadleftarrow C \rightarrowtail B$, $g = B \twoheadleftarrow E \rightarrowtail D$ in $Q\mathscr{C}$. Then

$$g \circ f = A \twoheadleftarrow C \twoheadleftarrow F \rightarrowtail E \rightarrowtail D,$$

where $F = C \times_B E$. Let $K_1 = \ker(E \to B) \cong \ker(F \to C)$, $K_2 = \ker(F \to A)$, and $K_3 = \ker(C \to A)$. Note that $K_1 \rightarrowtail K_2 \twoheadrightarrow K_3$ is exact. Now

$$L_{K_1} = [0 \rightarrowtail D]^{-1}[g][0 \rightarrowtail B], \quad L_{K_2} = [0 \rightarrowtail D]^{-1}[g \circ f][0 \rightarrowtail A], \quad L_{K_3} = [0 \rightarrowtail B]^{-1}[f][0 \rightarrowtail A],$$

so we have $[g] \cdot [f] = [g \circ f]$ as desired.

But now our presentation of $\pi_1(BQ\mathscr{C})$ agrees with that of $K_0(\mathscr{C})$, so we are done. ■

Given the above theorem, the following definition should not be unreasonable.

Definition 4. For an exact category $\mathscr{C}$, we define the K-theory space by $K(\mathscr{C}) := \Omega BQ\mathscr{C}$. The K-groups are then given by

$$K_i(\mathscr{C}) := \pi_i(K(\mathscr{C})) = \pi_{i+1}(BQ\mathscr{C}).$$

In the case of $\mathscr{C} = \mathcal{P}(R)$ one usually writes $K(R)$ for $K(\mathcal{P}(R))$. Also, in the case $\mathscr{C} = \mathcal{M}(R)$ for a noetherian ring R, one usually writes $G(R)$ for $K(\mathcal{M}(R))$. Note that the inclusion $\mathcal{P}(R) \subseteq \mathcal{M}(R)$ gives a map $K(R) \to G(R)$. The induced map on homotopy groups is usually referred to as the Cartan homomorphism.

1.4. Fundamental Theorems

Using the above definition of algebraic K-theory, Quillen was able to extend some of the fundamental theorems from K_0 and K_1 to the higher K-groups. We will see proofs later in the proseminar.

Theorem 2 (Resolution)**.** *Let $\mathcal{P} \subseteq \mathcal{M}$ be a full subcategory of an exact category which is closed under extension and kernels of $\mathcal{M}$-admissible epi's and such that for every $M \in \mathcal{M}$ there is a resolution $0 \to P_1 \to P_0 \to M \to 0$ with $P_0, P_1 \in \mathcal{P}$. Then $BQ\mathcal{P} \to BQ\mathcal{M}$ is a homotopy equivalence.*

Corollary 1. *The Cartan homomorphism $K_*(R) \to G_*(R)$ is an isomorphism for a regular ring R.*

Theorem 3 (Devissage)**.** *Let $\mathcal{A}$ be a small abelian category and $\mathcal{B} \subseteq \mathcal{A}$ be a full abelian subcategory such that the inclusion functor is exact. Suppose also that every object of $\mathcal{A}$ admits a finite filtration whose quotients are objects of $\mathcal{B}$. Then $BQ\mathcal{B} \to BQ\mathcal{A}$ is a homotopy equivalence.*

Theorem 4 (Localisation)**.** *Let $\mathcal{A}$ be a small abelian category, let $\mathcal{B}$ be a full abelian subcategory which is closed under subobjects, quotients, and extensions (this is called a Serre subcategory), and let $\mathcal{A}/\mathcal{B}$ be the quotient. Then $BQ\mathcal{B} \to BQ\mathcal{A} \to BQ\mathcal{A}/\mathcal{B}$ is a homotopy fiber sequence.*

Theorem 5 (Fundamental Theorem)**.** *For a Noetherian ring R, we have*

(1) $G_i(R) \cong G_i(R[t])$ *for all* $i \geq 0$
(2) $G_i(R[t, t^{-1}]) \cong G_i(R) \oplus G_{i-1}(R)$ *for all* $i \geq 0$ *(with* $G_{-1}(R) = 0$*).*

2. $+ = Q$

The main goal will be to prove

Theorem 6. *For a ring R,*

$$\Omega B(Q\mathcal{P}(R)) \simeq BGl(R)^+ \times K_0(R).$$

We will do this by introducing yet one more construction: the category $S^{-1}S$ associated to any symmetric monoidal category S. We will then prove

Theorem 7. *For $S = \mathrm{iso}\mathcal{P}(R)$, we have*

$$B(S^{-1}S) \simeq BGl(R)^+ \times K_0(R).$$

and

Theorem 8. *For $\mathscr{C}$ a split exact category and $S = \mathrm{iso}\mathscr{C}$, we have*

$$\Omega B(Q\mathscr{C}) \simeq B(S^{-1}S).$$

2.1. The $S^{-1}S$ construction

Suppose that S is a symmetric monoidal category. Then BS is an H-space, the product being given by $\oplus : S \times S \to S$. Moreover, the axioms for a symmetric monoidal category imply that BS is in fact homotopy-associative and homotopy-commutative (in fact BS is associate and commutative to up to all higher homotopies, so that is an E_∞-space).

Unfortunately, if a category has an inital object then its classifying space is contractible, so the above H-space will often be uninteresting. On the other hand, there is a way of obtaining an interesting H-space. Namely, let $\mathrm{iso}S$ be the subcategory of isomorphisms of S. That is, $\mathrm{iso}S$ has the same objects as S, but the morphisms are only the isomorphisms in S. Then $\mathrm{iso}S$ is still symmetric monoidal, and so $B(\mathrm{iso}S)$ is an H-space.

Example 2. (a) Any additive category is symmetric monoidal, with monoidal product given by the direct sum.

(b) As an example of (a), the category $\mathcal{F}(R)$ of finitely generated free modules over R is symmetric monoidal, with product given by $\oplus$. We have

$$B(\mathrm{iso}\mathcal{F}(R)) \cong \coprod_M B(Aut(M)) \cong \coprod_n BGl_n(R),$$

where the coproduct runs over isomorphism classes of finitely generated free modules M.

(c) The category $\mathcal{P}(R)$ of finitely generated projective modules over R is symmetric monoidal, with product given by $\oplus$. We have

$$B(\mathrm{iso}\mathcal{P}(R)) \cong \coprod B(Aut(P)),$$

where the coproduct runs over isomorphism classes of finitely generated projective modules P.

As before, however, the space $B(\text{iso}S)$ is not quite the right space–we need to apply some sort of group completion.

Definition 5. Let S be a symmetric monoidal category. We define a new category $S^{-1}S$ as follows. The objects of $S^{-1}S$ are pairs (m, n) of objects in S. A morphism $(m, n) \to (p, q)$ in $S^{-1}S$ is an equivalence class of triples

$$(r, r \oplus m \xrightarrow{f} p, r \oplus n \xrightarrow{g} q)$$

where a triple of this form is said to be equivalent to a triple

$$(r', r' \oplus m \xrightarrow{f'} p, r' \oplus n \xrightarrow{g'} q)$$

if there is an isomorphism $r \cong r'$ making the relevant diagrams commute.

Composition is defined as follows: given a pair of morphisms

$$(r, r \oplus m \xrightarrow{f} p, r \oplus n \xrightarrow{g} q),$$

and

$$(s, s \oplus p \xrightarrow{\varphi} u, s \oplus q \xrightarrow{\psi} v),$$

the composite is defined as

$$(s \oplus r, s \oplus r \oplus m \xrightarrow{\varphi \circ (s \oplus f)} u, s \oplus r \oplus n \xrightarrow{\psi (s \oplus g)} v.$$

Remark 3. Note that $S^{-1}S$ is symmetric monoidal with $(m, n) \oplus (p, q) = (m \oplus p, n \oplus q)$. Moreover, we have a (strict) monoidal functor $S \to S^{-1}S$ given by $m \mapsto (m, 0)$, where 0 is the unit of S. This induces a map $BS \to B(S^{-1}S)$ of H-spaces and a map of abelian monoids

$$\pi_0(BS) \to \pi_0(B(S^{-1}S)).$$

In fact $\pi_0(B(S^{-1}S))$ is an abelian group and the above map is a group completion (the inverse in π_0 of an element (m, n) is (n, m)).

Definition 6. Let S be a symmetric monoidal groupoid. The K-theory space $K(S)$ of S is then defined to be $B(S^{-1}S)$. For a general symmetric monoidal category S, we define the K-theory space of S to be $K(\text{iso}S)$.

As usual, the K-groups of S are simply the homotopy groups of the K-theory space.

As we have seen above, $\pi_0(B(S^{-1}S))$ is the group completion of $\pi_0(B(S))$, so $K_0(\mathcal{P}(R)) = K_0(R)$ as defined classsically.

2.2. $+ = S^{-1}S$

Definition 7. A *group completion* of a homotopy associative, homotopy commutative H-space X is a map $X \xrightarrow{\varphi} Y$ where Y is again a homotopy associative, homotopy commutative H-space such that

(1) $\varphi_* : \pi_0(X) \to \pi_0(Y)$ is a group completion of the commutative monoid $\pi_0(X)$ and
(2) the induced map $\varphi_* : [\pi_0(X)]^{-1} H_*(X) \to H_*(Y)$ is an isomorphism.

Theorem 9. *(Quillen) If S is a symmetric monoidal groupoid such that*

$$\text{Aut}(s) \to \text{Aut}(s \oplus t)$$

is injective for all $s, t \in S$ (we say translations are faithful in S), then $BS \to B(S^{-1}S)$ is a group completion.

Proof. We have already verified the condition on π_0, so it remains to compute the homology of $B(S^{-1}S)$ and verify that the map gives an isomorphism.

We begin by defining the "translation category" ES as follows: the objects are the objects of S, and a morphism $s \to t$ in ES is given by isomorphism classes of pairs

$$(r, r \oplus s \to t),$$

where again such a pair is isomorphic to

$$(r', r' \oplus s \to t)$$

if we have an isomorphism $r \cong r'$ making the appropriate diagram commute.

There is an obvious functor $P : S^{-1}S \to ES$ (in fact there are two) given by $(s,t) \mapsto t$. To compute the homology of $BS^{-1}S$, we will use the same ideas that go into the proofs of Quillen's Theorems A and B. Let $G : (S^{-1}S)^{op} \times ES \to \text{Set}$ be defined by

$$G((r,s),t) = \text{Hom}_{ES}(P(r,s),t).$$

Then note that for fixed $t \in ES$

$$B_\bullet(G(-,t), S^{-1}S, *) = B_\bullet(P \downarrow d).$$

Similarly, for fixed $(r,s) \in S^{-1}S$ we have

$$B_\bullet(*, ES, G((r,s),-)) = B_\bullet(P(r,s) \downarrow ES).$$

But since $P(r,s)$ is initial in $(P(r,s) \downarrow ES)$, this latter simplicial set is contractible.

Now recall that for any functor $G : \mathscr{C}^{op} \times \mathscr{D} \to \text{Set}$ we have the double two-sided bar construction given in bidegree (p,q) by

$$B_{p,q}(*, \mathscr{D}, G, \mathscr{C}, *) = \coprod_{d_0 \to \cdots \to d_p} \coprod_{c_0 \to \cdots \to c_q} G(c_q, d_0)$$

where the coproducts run over elements in $B_p\mathscr{D}$ and $B_q\mathscr{C}$, respectively. In our situation, we have

$$\begin{aligned} B_\bullet\big(*, ES, B_\bullet(G(-,-), S^{-1}S, *)\big) &\cong B_{\bullet,\bullet}(*, ES, G, S^{-1}S, *) \\ &\cong B_\bullet\big(B_\bullet(*, ES, G(-,-)), S^{-1}S, *\big) \\ &\simeq B_\bullet(*, S^{-1}S, *) = B_\bullet(S^{-1}S) \end{aligned}$$

We can now hit this bisimplicial set with the free abelian group functor and take the alternating sum of face maps to get a double complex. The above analysis shows that taking homology first in the p direction and then in the q direction gives $H_*(B(S^{-1}S))$.

Taking homology in the other order gives us a spectral sequence with

$$E^0_{p,q} = \bigoplus_{\substack{t_0 \to \cdots \to t_p \\ \in B_p(ES)}} \quad \bigoplus_{\substack{(r_0,s_0) \to \cdots \to (r_q,s_q) \\ \in B_q(P \downarrow t_0)}} \mathbb{Z}$$

converging to $H_*(B(S^{-1}S))$. We see that

$$E^1_{p,q} = \bigoplus_{\substack{t_0 \to \cdots \to t_p \\ \in B_p(ES)}} H_q(B(P \downarrow t_0)).$$

Let us take a moment to analyze these fibers $B(P \downarrow t_0)$.

Lemma 2. *For each $t_0 \in ES$, the inclusion $P^{-1}(t_0) \hookrightarrow (P \downarrow t_0)$ has a left adjoint I.*

Proof. This will use all of our assumptions about S. The point is that our assumptions about S imply that given a morphism $(u, u \oplus s \to t)$ from s to t in ES, the u is determined up to *canonical* isomorphism (it is by definition defined up to isomorphism). The right adjoint is then defined by

$$I : (P(r, s) \to t_0) \mapsto (u \oplus r, t_0)$$

where u is a choice of representative for the given morphism $P(r, s) = s \to t_0$. The previous statement imply that any set of choices will be compatible. $\square$

This implies that $B(P^{-1}(t_0)) \to B(P \downarrow t_0)$ is a homotopy equivalence for all $t_0 \in ES$. Moreover, $P^{-1}(-)$ now determines a *functor* on ES: given a morphism $\alpha = (u, u \oplus t_0 \to t_1)$ in ES, we define a functor $P^{-1}(t_0) \to P^{-1}(t_1)$ by

$$P^{-1}(t_0) \hookrightarrow (P \downarrow t_0) \xrightarrow{\alpha_*} (P \downarrow t_1) \xrightarrow{I} P^{-1}(t_1).$$

In fact, we see that this functor is defined by $(r, t_0) \mapsto (u \oplus r, t_1)$.

Finally, note that projection onto the first coordinate gives an isomorphism of categories $P^{-1}(t_0) \to S$ for every $t_0 \in ES$. Again, a morphism $\alpha = (u, u \oplus t_0 \to t_1)$ corresponds, under these isomorphisms, to translation by u on S.

The above discussion allows us to identify the E^1 term of our spectral sequence as

$$E^1_{p,q} = \bigoplus_{\substack{t_0 \to \cdots \to t_p \\ \in B_p(ES)}} H_q(BS).$$

For fixed q, this looks like the chain complex whose homology is the homology of $B(ES)$ with coefficients in the local system $H_q(BS)$, but $H_q(BS)$ does not give a local coefficient system. The point is that a morphism $(u, u \oplus t_0 \to t_1)$ in ES induces an endomorphism of $H_q(BS)$; this endomorphism, as we have seen, is given by translation by u. But translation by u need not induce a homotopy equivalence on BS.

If we localize with respect to the multiplicatively closed set $\pi_0(BS) \subseteq H_*(BS)$, then each translation *will* induce a homotopy equivalence on BS, and $[\pi_0(BS)]^{-1}H_*(BS)$ does define a local coefficient system on $B(ES)$. Since localization is exact, we now have a spectral sequence with E^1 term given by

$$E^1_{p,q} = \bigoplus_{\substack{t_0 \to \cdots \to t_p \\ \in B_p(ES)}} [\pi_0(BS)]^{-1}H_q(BS)$$

converging to

$$[\pi_0(BS)]^{-1}H_*(B(S^{-1}S)) \cong H_*(B(S^{-1}S)).$$

But now the E^2 term is given by

$$E^2_{p,q} = H_p\big(B(ES), H_q(BS)\big).$$

In fact, 0 is initial in ES, so $B(ES)$ is contractible and we have

$$E^2_{p,q} = \begin{cases} [\pi_0(BS)]^{-1}H_q(BS) & p = 0 \\ 0 & p > 0. \end{cases}$$

Thus the spectral sequence collapses at E^2 and we get the desired isomorphism

$$[\pi_0(BS)]^{-1}H_*(BS) \cong H_*(B(S^{-1}S)).$$

■

Corollary 2. *Let $S = \mathrm{iso}\mathcal{F}(R)$, so that $B(S) = \coprod_n Gl_n(R)$. Then*

$$B(S^{-1}S) \simeq \mathbb{Z} \times BGl(R)^+.$$

Proof. Let $B_0 \subseteq B(S^{-1}S)$ be the component of $(0,0)$. Then $B(S^{-1}S) \simeq \mathbb{Z} \times B_0$, so it suffices to show $B_0 \simeq BGl(R)^+$. It suffices to construct an acyclic map $\varphi : BGl(R) \to B_0$.

We first define $\varphi_n : BGl_n(R) \to B\,\mathrm{Aut}(R^n, R^n) \hookrightarrow B_0$ to be the map induced by the homomorphism $Gl_n(R) \to \mathrm{Aut}(R^n, R^n)$ given by $g \mapsto (g, 1)$. Then the following diagram commutes for every n:

$$\begin{tikzcd}
BGl_n(R) \arrow[r, "\varphi_n"] \arrow[d, "\oplus R"] & B_0 \arrow[d, "\oplus(R,R)"] \\
BGl_{n+1}(R) \arrow[r, "\varphi_{n+1}"] & B_0.
\end{tikzcd}$$

Now $B_0 \xrightarrow{\oplus(R,R)} B_0$ is homotopic to the identity, so passing to hocolim's, we get $BGl(R) \to B_0$. To see that this induces an isomorphism on homology, we use that $B(S^{-1}S)$ is a group completion of $B(S)$:

$$\begin{aligned}
H_*(B(S^{-1}S)) \cong H_*(BS)\left[\frac{1}{R}\right] &\cong \mathrm{colim}\big(H_*(BS) \xrightarrow{(\oplus R)_*} H_*(BS) \xrightarrow{(\oplus R)_*} H_*(BS) \to \dots\big) \\
&\cong H_*(\mathrm{colim}\big(BS \xrightarrow{\oplus R} BS \xrightarrow{\oplus R} BS \to \dots\big))
\end{aligned}$$

From this description, it is clear that $\varphi : BGl(R) \to B_0$ induces an isomorphism on homology, and this gives $BGl(R)^+ \simeq B_0$. ∎

Definition 8. A monoidal functor $F : S \to T$ is *cofinal* if for every $t \in T$ there is $t_2 \in T$ and $s \in S$ such that $t \oplus t_2 \cong F(s)$.

Corollary 3 (Cofinality)**.** *If $F : S \to T$ is cofinal and $\mathrm{Aut}_S(s) \cong \mathrm{Aut}_T(F(s))$ for all $s \in S$, then the map $B(S^{-1}S) \to B(T^{-1}T)$ induces an equivalence of the basepoint components.*

Proof. This follows easily from the description of $B(S^{-1}S)$ and $B(T^{-1}T)$ as group completions. We will write $B_0(S^{-1}S)$ and $B_0(T^{-1}T)$ for the basepoint components. We have

$$\begin{aligned}
H_*(B_0(S^{-1}S)) \cong \operatorname*{colim}_{ES} H_*(B\,\mathrm{Aut}_S(s)) &\cong \operatorname*{colim}_{ES} H_*(B\,\mathrm{Aut}_T(F(s))) \\
&\cong \operatorname*{colim}_{ET} H_*(B\,\mathrm{Aut}_T(t)) \cong H_*(B_0(T^{-1}T))
\end{aligned}$$

where the colimits are taken over the appropriate translation categories and the identification from the first line to the second comes from cofinality. Since $B_0(S^{-1}S)$ and $B_0(T^{-1}T)$ are connected H-spaces, this implies that they are equivalent. ∎

We now get our desired comparison

Theorem 10. *Let $S = \mathrm{iso}\mathcal{P}(R)$. Then*

$$B(S^{-1}S) \simeq K_0(R) \times BGl(R)^+.$$

Proof. This follows from Remark 3, Corollary 2, and Corollary 3. ∎

2.3. $S^{-1}S = Q$

Let $\mathscr{C}$ be an exact category and $S = \mathrm{iso}\mathscr{C}$, considered as a symmetric monoidal category under $\oplus$.

Definition 9. We define a new category $\mathcal{E}\mathscr{C}$, the category of exact sequences in $\mathscr{C}$, as follows. The objects are the exact sequences of $\mathscr{C}$. The set of morphisms from $A \rightarrowtail B \twoheadrightarrow C$ to $A' \rightarrowtail B' \twoheadrightarrow C'$ is given by equivalence classes of diagrams

$$\begin{tikzcd}
A \arrow[r, tail] & B \arrow[r, two heads] & C \\
A' \arrow[u, tail, "\alpha"] \arrow[r, tail] \arrow[d, equal] & B \arrow[u, equal] \arrow[r, two heads] \arrow[d, tail, "\beta"] & C'' \arrow[u, two heads] \arrow[d, tail] \\
A' \arrow[r, tail] & B' \arrow[r, two heads] & C',
\end{tikzcd}$$

where two such diagrams are equivalent if there is an isomorphism between them which is the identity except at C''.

Remark 4. This forces the bottom right square to be a pullback square.

Note that $T(A \hookrightarrow B \twoheadrightarrow C) = C$ defines a functor $\mathcal{E}\mathscr{C} \to Q\mathscr{C}$. We also have a functor $S = \mathrm{iso}\mathscr{C} \to \mathcal{E}\mathscr{C}$ which sends an object C to $C \overset{\mathrm{id}}{\rightarrowtail} C \twoheadrightarrow 0$. Note that the functor $S \times \mathcal{E}\mathscr{C} \to \mathcal{E}\mathscr{C}$ given by

$$(s, (A \rightarrowtail B \twoheadrightarrow C)) \mapsto (s \oplus A \rightarrowtail s \oplus B \twoheadrightarrow C)$$

defines an "action" of S on $\mathcal{E}\mathscr{C}$.

Definition 10. If S acts on $\mathscr{D}$, we can define a new category $S^{-1}\mathscr{D}$ whose objects are given by pairs $(s, d) \in S \times \mathscr{D}$. A morphism $(s_1, d_1) \to (s_2, d_2)$ is given by an equivalence class of triples

$$(t, (t \oplus s_1, t \oplus d_1) \xrightarrow{f,g} (s_2, d_2)),$$

where such a triple is equivalent to

$$(r, (r \oplus s_1, r \oplus d_1) \xrightarrow{f',g'} (s_2, d_2))$$

if there is an isomorphism $t \cong r$ making the relevant diagram commute.

Remark 5. Note that the proof of Theorem 9 generalizes easily to give that if S is as in the theorem and S acts on some category $\mathscr{D}$ then

$$[\pi_0(BS)]^{-1} H_*(B\mathscr{D}) \cong H_*(S^{-1}B\mathscr{D}).$$

Finally, since the functor $T : \mathcal{E}\mathscr{C} \to Q\mathscr{C}$ defined earlier satisfies

$$T(s \oplus (A \rightarrowtail B \twoheadrightarrow C)) = T(A \rightarrowtail B \twoheadrightarrow C),$$

we get an induced functor $\mathscr{T} : S^{-1}\mathcal{E}\mathscr{C} \to Q\mathscr{C}$.

Proposition 1. *For each $C \in Q\mathscr{C}$, the inclusion $\mathscr{T}^{-1}(C) \hookrightarrow (C \downarrow \mathscr{T})$ has a right adjoint, so that $B(\mathscr{T}^{-1}(C)) \simeq B(C \downarrow \mathscr{T})$.*

Proof. We will define the functor on objects. Let

$$C \to \mathscr{T}(s, (A' \rightarrowtail B' \twoheadrightarrow C')) = C'$$

be in $(C \downarrow \mathscr{T})$. Suppose the morphism $C \to C'$ in $Q\mathscr{C}$ is given by

$$C \twoheadleftarrow C'' \rightarrowtail C'$$

and form the pullback

$$\begin{tikzcd} B \arrow[r, two heads] \arrow[d, tail] & C'' \arrow[d, tail] \\ B' \arrow[r, two heads] & C'. \end{tikzcd}$$

Since $B \twoheadrightarrow C'' \twoheadrightarrow C$ is an admissible epimorphism, it has a kernel in $\mathscr{C}$, and we have an exact sequence

$$A \rightarrowtail B \twoheadrightarrow C.$$

Our functor then takes

$$C \to \mathscr{T}(s, (A' \rightarrowtail B' \twoheadrightarrow C'))$$

to

$$(s, (A \rightarrowtail B \twoheadrightarrow C)).$$

It is then clear how to define it on morphisms and straightforward to check that this in fact defines a right adjoint to the inclusion $\mathscr{T}^{-1}(C) \hookrightarrow (C \downarrow \mathscr{T})$. ∎

Proposition 2. *Suppose $\mathscr{C}$ is split exact and define $S \to T^{-1}(C)$ by $A \mapsto (A \rightarrowtail A \oplus C \twoheadrightarrow C)$. Then the induced map $B(S^{-1}S) \to B(\mathscr{T}^{-1}(C)) = B(S^{-1}T^{-1}(C))$ is an equivalence.*

Theorem 11. *Let $\mathscr{C}$ be a split exact category and $S = \mathrm{iso}\mathscr{C}$. Then the sequence*

$$B(S^{-1}S) \to B(S^{-1}\mathcal{E}\mathscr{C}) \xrightarrow{B\mathscr{T}} B(Q\mathscr{C})$$

is a quasifibration sequence.

Proof. We will use Quillen's Theorem B. It is clear that every comma category $(C \downarrow \mathscr{T})$ is nonempty. It remains to check that for any morphism $f : C \to C'$ in $Q\mathscr{C}$, the induced map $B(C' \downarrow \mathscr{T}) \to B(C \downarrow \mathscr{T})$ is a homotopy equivalence. It suffices to do this for morphisms of the form $0 \rightarrowtail C$ and $0 \twoheadleftarrow C$. We use Proposition 1, and note that $\mathscr{T}^{-1}(0) \simeq S^{-1}S$.

In the first case, we see that the composition of the equivalence $B(S^{-1}S) \to B(\mathscr{T}^{-1}(C))$ of Proposition 2 with $f^* : B(\mathscr{T}^{-1}(C)) \to B(S^{-1}S)$ is the identity. It follows that f^* is an equivalence.

When f is $0 \twoheadleftarrow C$, the same composition is the endofunctor $(A, B) \mapsto (A, B \oplus C)$ on $S^{-1}S$. Of course this induces an equivalence on $B(S^{-1}S)$ ($(A, B) \mapsto (A \oplus C, B)$ is a homotopy inverse). ∎

Theorem 12. *If $\mathscr{C}$ is split exact, then $\Omega BQ\mathscr{C} \simeq B(S^{-1}S)$.*

Proof. It remains to show, by the previous theorem, that $B(S^{-1}\mathcal{E}\mathscr{C})$ is contractible. First, $\mathcal{E}\mathscr{C}$ is contractible. We can see this by applying Quillen's Theorem A to the functor $m : \mathcal{E}\mathscr{C} \to iQ\mathscr{C}$ defined by $m(A \rightarrowtail B \twoheadrightarrow C) = B$, where $iQ\mathscr{C}$ is the subcategory of $Q\mathscr{C}$ consisting of admissible monomorphisms. Theorem A tells us that this is a homotopy equivalence ($0 \rightarrowtail B \twoheadrightarrow B$ is terminal in $(m \downarrow B)$), but $iQ\mathscr{C}$ has an initial object and so is contractible.

But then S certainly acts by homotopy equivalences on $\mathcal{E}\mathscr{C}$. It follows by Remark 5 that $B(\mathcal{E}\mathscr{C}) \to B(S^{-1}\mathcal{E}\mathscr{C})$ is an equivalence. ∎