Algebra Prelim

January 2008

Provide proofs for all statements, citing any theorems that may be needed.
If necessary, you may use the results from other parts of this test, even though you may not have successfully proved them.

1. Let F be a field and let U be the vector space of all 3×3 matrices
$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$
over F. Let $S \subset U$ consist of all such matrices where $\alpha_{11} = \alpha_{12}$ and where $\alpha_{23} = \alpha_{33}$. Show that S is a subspace of U and find its dimension.

2. Let G be a group of order 45. Show the following:
   (a) G contains a normal subgroup of order 5 and a normal subgroup of order 9.
   (b) G is abelian.

3. Consider the abelian group $\mathbf{Z}/2\mathbf{Z}$. Find out how many homomorphisms there are of this group into the group $S_3 \times S_3$ (S_3 is the symmetric group).

4. Let G be a finite group that contains exactly 3 subgroups (including the trivial ones). Show that $|G| = p^2$ for some prime p and that G is cyclic.

5. Let R be an integral domain. Then:
   (a) If $a \in R$ is not a unit and $a \neq 0$ show that the ideal (a, x) of the polynomial ring $R[x]$ is not principal.
   (b) Show that $R[x]$ is a principal ideal domain if and only if R is a field.

6. Let $\mathbf{Q}$ denote the field of rational numbers. Show that the following polynomials are irreducible in the given rings of polynomials:
   (a) $f = x^7 - 8x^4 + 2x^2 - 4x + 2 \in \mathbf{Q}[x]$
   (b) $g = 2y^2x^4 + y^3x^3 + 5yx^2 + x^3 + 2 \in \mathbf{Q}[x, y]$
   (c) $h = 5x^3 + 8x^2 - 3x + 4 \in \mathbf{Q}[x]$

7. (a) Find the minimal polynomial f of $a = \sqrt[3]{5} + 2$ over $\mathbf{Q}$.
   (b) Compute the Galois group of f over $\mathbf{Q}$ up to isomorphism.

8. (a) Determine the field extension degree $[\mathbf{Q}(\sqrt{2})(i) : \mathbf{Q}(\sqrt{2})]$
(b) Show that $\mathbf{Q}(\sqrt{2}+i) = \mathbf{Q}(\sqrt{2})(i)$.
(c) Compute the minimal polynomial of $\sqrt{2}+i$ over $\mathbf{Q}$.

9. Consider the polynomial $f = x^4 + 1 \in \mathbf{Q}[x] \subset \mathbf{R}[x]$ where $\mathbf{R}$ denotes the field of real numbers. Then:

   (a) Factor f into irreducible polynomials in $\mathbf{Q}[x]$ as well as into irreducible polynomials in $\mathbf{R}[x]$.

   (b) Find the Galois groups of f over $\mathbf{Q}$ and over $\mathbf{R}$.