Algebra Prelim

January 11, 2010

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

1. Let V be a finite-dimensional vector space over a field k and let T be a linear transformation from V to V. Suppose that the images of T and T^2 have the same dimension, i.e., $\text{rank}(T) = \text{rank}(T^2)$. Prove that the image and the kernel of T are disjoint, i.e., have only the zero common vector.
(We recall that T^2 denotes the composition of T with itself.)

2. Let V be a finite-dimensional $\mathbb{R}$-vector space, and let T be a non-trivial linear transformation from V to V. Show that if $T^3 = -T$, then either T has no real eigenvalues or 0 is the unique real eigenvalue of T. Furthermore, show that both cases do occur.

3. Let G be a finite cyclic group of order n with generator a. Prove that a^i has order $n/\gcd(i, n)$ for all $i \geq 1$.

4. Let G and H be groups. Let $\varphi\colon G \longrightarrow H$ be a surjective homomorphism and let K denote the kernel of φ. For $h \in H$ let $\varphi^{-1}(h) = \{g \in G \,|\, \varphi(g) = h\}$ be the fiber of h. Show that for all $h \in H$ we have

$$\varphi^{-1}(h) = \widetilde{g}K = K\widetilde{g},$$

where $\widetilde{g}$ is any element of $\varphi^{-1}(h)$.

5. Let G be a simple group of order 168. How many elements of order 7 does G have?

6. An *idempotent* of a ring with identity is an element e such that $e^2 = e$. Let T be a commutative ring with identity and let $e \in T$ be an idempotent. Prove that

(a) $f = 1 - e$ is an idempotent,
(b) $R = Te$ and $S = Tf$ are rings,
(c) $T \simeq R \times S$.

7. Let R be a finite commutative ring with identity. Prove that every prime ideal of R is a maximal ideal.

8. Factor the following (possibly irreducible) polynomials into their irreducible factors in the given polynomial ring:

(a) $3X^3 - 3X^2 - 3X - 6 \in \mathbb{Z}[X]$;

(b) $X^4 + 1 \in (\mathbb{Z}/2\mathbb{Z})[X]$;

(c) $X^7 - 4 - i \in \mathbb{Q}(i)[X]$;

9. Determine the splitting field $E \subset \mathbb{C}$ of $X^4 - 7X^2 + 10$ over $\mathbb{Q}$ and its automorphism group. Be sure to specify all the maps.

10. Let E be a splitting field of an irreducible and separable polynomial $f \in K[X]$ over the field K. Assume that the Galois group of E/K is abelian, and let $\alpha \in E$ be a root of f. Show that $E = K(\alpha)$ and $[E : K] = \deg f$.