Algebra Prelim

January 5, 2011

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

1. (a) Let V be a 3-dimensional vector space over a field K with basis $\{v_1, v_2, v_3\}$. Find a condition on the characteristic of K that guarantees that $\{v_1 + v_2, v_2 + v_3, v_3 + v_1\}$ is also a basis of V.

(b) If V is a 4-dimensional vector space with basis $\{v_1, v_2, v_3, v_4\}$, show that $\{v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4 + v_1\}$ is not a basis of V.

(c) Conjecture a generalization to the case when $\{v_1 + v_2, v_2 + v_3, \ldots, v_n + v_1\}$ is a basis of V. In other words, for which n and which characteristic of K is $\{v_1 + v_2, v_2 + v_3, \ldots, v_n + v_1\}$ a basis of V (where $\{v_1, \ldots, v_n\}$ is a basis of V)? (You do not need to prove your conjecture.)

2. Let $f : V \to V$ be an endomorphism (linear transformation) of a finite-dimensional vector space V such that $f \circ f = f$, $\ker f \neq \{0\}$, and $\operatorname{im} f \neq \{0\}$.

(a) Find the eigenvalues of f.

(b) Show that f is diagonalizable and describe its diagonal matrix representation as closely as possible.

3. Let g be an element of a group G. If there are exactly two conjugates of g in G, show that either there are exactly two conjugates of g^2 in G or g^2 is in the center of G.

4. Let G be a cyclic group of order 168 that is generated by a. Consider the group homomorphism $\varphi : G \to G$, defined by $\varphi(a) = a^{105}$. Determine the order of $\ker \varphi$ and $\operatorname{im} \varphi$.

5. Find a generator of the ideal $I = (85, 1 + 13i)$ of $\mathbb{Z}[i] \subset \mathbb{C}$.

6. Let R be a factorial subring (UFD) of $\mathbb{C}$ that does not contain $\sqrt{-5}$. Show that the rings $R[\sqrt{-5}]$ and $R[X]/(X^2+5)$ are isomorphic. (Notice that R is not necessarily a field.)

7. Let R be a commutative ring with identity 1, and let I, J be ideals of R such that $I + J = R$. Without using the Chinese Remainder Theorem show:

(a) $IJ = I \cap J$.

(b) There are elements $a \in I$ and $b \in J$ such that $a \equiv 1 \bmod J$ and $b \equiv 1 \bmod I$.

(c) The rings R/IJ and $R/I \oplus R/J$ are isomorphic.

8. Consider $\alpha := \sqrt{2+\sqrt{3}} \in \mathbb{R}$.

(a) Determine the minimal polynomial f of α over $\mathbb{Q}$.

(b) Show that f splits over $\mathbb{Q}(\alpha)$.

(c) Determine abstractly the Galois group of f over $\mathbb{Q}$.

9. Let E/K be a Galois extension of degree 55 whose Galois group is not abelian. Determine the number of intermediate fields L with $K \subset L \subset E$ and $[L : K] = 11$.