Algebra Prelim
January 5, 2011

• Provide proofs for all statements, citing theorems that may be needed.
• If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.
• Do as many problems as you can and present your solutions as carefully as possible.

Good luck!

1. (a) Let $V$ be a 3-dimensional vector space over a field $K$ with basis $\{v_1, v_2, v_3\}$. Find a condition on the characteristic of $K$ that guarantees that $\{v_1 + v_2, v_2 + v_3, v_3 + v_1\}$ is also a basis of $V$.

(b) If $V$ is a 4-dimensional vector space with basis $\{v_1, v_2, v_3, v_4\}$, show that $\{v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4 + v_1\}$ is not a basis of $V$.

(c) Conjecture a generalization to the case when $\{v_1 + v_2, v_2 + v_3, \ldots, v_n + v_1\}$ is a basis of $V$. In other words, for which $n$ and which characteristic of $K$ is $\{v_1 + v_2, v_2 + v_3, \ldots, v_n + v_1\}$ a basis of $V$ (where $\{v_1, \ldots, v_n\}$ is a basis of $V$)? (You do not need to prove your conjecture.)

2. Let $f : V \to V$ be an endomorphism (linear transformation) of a finite-dimensional vector space $V$ such that $f \circ f = f$, $\ker f \neq \{0\}$, and $\operatorname{im} f \neq \{0\}$.

(a) Find the eigenvalues of $f$.

(b) Show that $f$ is diagonalizable and describe its diagonal matrix representation as closely as possible.

3. Let $g$ be an element of a group $G$. If there are exactly two conjugates of $g$ in $G$, show that either there are exactly two conjugates of $g^2$ in $G$ or $g^2$ is in the center of $G$.

4. Let $G$ be a cyclic group of order 168 that is generated by $a$. Consider the group homomorphism $\varphi : G \to G$, defined by $\varphi(a) = a^{105}$. Determine the order of $\ker \varphi$ and $\operatorname{im} \varphi$.

5. Find a generator of the ideal $I = (85, 1 + 13i)$ of $\mathbb{Z}[i] \subset \mathbb{C}$.
6. Let $R$ be a factorial subring (UFD) of $\mathbb{C}$ that does not contain $\sqrt{-5}$. Show that the rings $R[\sqrt{-5}]$ and $R[X]/(X^2 + 5)$ are isomorphic. (Notice that $R$ is not necessarily a field.)

7. Let $R$ be a commutative ring with identity 1, and let $I, J$ be ideals of $R$ such that $I + J = R$. Without using the Chinese Remainder Theorem show:
   (a) $IJ = I \cap J$.
   (b) There are elements $a \in I$ and $b \in J$ such that $a \equiv 1 \mod J$ and $b \equiv 1 \mod I$.
   (c) The rings $R/IJ$ and $R/I \oplus R/J$ are isomorphic.

8. Consider $\alpha := \sqrt{2} + \sqrt{3} \in \mathbb{R}$.
   (a) Determine the minimal polynomial $f$ of $\alpha$ over $\mathbb{Q}$.
   (b) Show that $f$ splits over $\mathbb{Q}(\alpha)$.
   (c) Determine abstractly the Galois group of $f$ over $\mathbb{Q}$.

9. Let $E/K$ be a Galois extension of degree 55 whose Galois group is not abelian. Determine the number of intermediate fields $L$ with $K \subset L \subset E$ and $[L : K] = 11$. 

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