

Algebra Prelim

January 5, 2012

- Provide proofs for all statements, citing theorems that may be needed.
- If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.
- Do as many problems as you can and present your solutions as carefully as possible.

Good luck!

- (1) Let V be a finite-dimensional vector space and $T : V \rightarrow V$ be a linear map.
- a) Show that $\ker T^j \subseteq \ker T^{j+1}$ for all $j \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ and conclude that there exists some $k \in \mathbb{N} := \{1, 2, 3, \dots\}$ such that $\ker T^j = \ker T^k$ for all $j \geq k$.
 - b) Show that $\ker T^k \cap \operatorname{im} T^k = \{0\}$ where k is as in (a).
- (2) Let V be a two-dimensional vector space and $F : V \rightarrow V$ be a linear map. Suppose F has exactly one eigenvalue, denoted by λ . Show that $F(v) - \lambda v$ is contained in $\operatorname{eig}(\lambda; F)$ for all $v \in V$, where $\operatorname{eig}(\lambda; F)$ is the eigenspace of F to the eigenvalue λ .
- (3) Let G be a group with $5^2 \cdot 7$ elements.
- a) Show that G is abelian.
 - b) Show that if G does not contain an element of order 25, then it contains an element of order 35.
- (4) Let G be a group of order p^r for some prime p and $r \in \mathbb{N}$. Let X be the set of all subgroups of G and consider the group action $G \times X \rightarrow X$, $(g, H) \mapsto gHg^{-1}$. Furthermore, let $m = |X|$ (thus, m is the number of all subgroups of G), and let n be the number of all normal subgroups of G . Show that p divides $m - n$.
[Hint: A subgroup H is normal in G if and only if its orbit \mathcal{O}_H satisfies $\mathcal{O}_H = \{H\}$.]
- (5) Show that the kernel of the substitution homomorphism $\psi : \mathbb{Z}[x] \rightarrow \mathbb{Q}$, $f \mapsto f(\frac{1}{2})$ is a principal ideal.
- (6) a) Let R be a domain. Prove that any prime element in R is irreducible.
b) Show that the element $4 + \sqrt{10}$ is irreducible in the ring $\mathbb{Z}[\sqrt{10}]$, but not prime.
- (7) Let F be a finite field and let $f \in F[x]$ be a polynomial such that f and its derivative f' are relatively prime. Show that there exists an $n \in \mathbb{N}$ such that $f \mid (x^n - x)$.

(please turn over)

For the following problems recall that for a field extension $L \subseteq K$ the notation $\text{Aut}(K | L)$ denotes the group of all automorphisms of K that leave the elements of L fixed.

- (8) a) Show that if $\mathbb{Q} \subseteq K$ is a field extension of degree 2, then $|\text{Aut}(K | \mathbb{Q})| = 2$.
b) Show that for any odd number $n \in \mathbb{N}$ there exists a field extension $\mathbb{Q} \subseteq K$ of degree n such that $|\text{Aut}(K | \mathbb{Q})| = 1$.
- (9) Let $\zeta \in \mathbb{C}$ be a 31st primitive root of unity. Show that there exist exactly 8 distinct subfields L of $\mathbb{Q}(\zeta)$ (that is, 8 distinct subfields L satisfying $\mathbb{Q} \subseteq L \subseteq \mathbb{Q}(\zeta)$, including the two trivial ones).
- (10) Consider the polynomial $f := (x^2 - 2)(x^3 - 3) \in \mathbb{Q}[x]$, and let K be a splitting field of f over \mathbb{Q} .
a) Determine the degree of the field extension $\mathbb{Q} \subseteq K$.
b) Determine the isomorphism type of $\text{Aut}(K | \mathbb{Q})$.