Algebra Prelim

January 7, 2015

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

(1) Let V be a vector space over $\mathbb{R}$ of dimension $n > 1$. Let $(e_1, \ldots, e_n)$ be a basis of V. Consider the linear map $T : V \longrightarrow V$ defined by

$$T(e_i) = e_i + ie_n \text{ for } 1 \leq i \leq n-1 \text{ and } T(e_n) = 0.$$

Let A denote the matrix representation of T in the chosen basis.

a) Give the matrix A.

b) Determine the characteristic polynomial, the eigenvalues and eigenspaces of A.

c) Prove or disprove that A is diagonalizable.

(2) Let K be a field and $A, B \in K^{n\times n}$ be matrices such that $AB = BA$. Suppose all eigenspaces of A and B are 1-dimensional. Show that A and B have the same eigenvectors.

(3) Let G be a cyclic group of order n, written multiplicatively. Furthermore, let $m \in \mathbb{N}$ and $d = \gcd(m, n)$.

a) For $t \in \mathbb{N}$ define $\varphi_t : G \longrightarrow G,\ x \longmapsto x^t$. Show

$$\operatorname{im}(\varphi_m) = \operatorname{im}(\varphi_d) \text{ and } \ker(\varphi_m) = \ker(\varphi_d).$$

b) Let b be any element of G. Show that the equations $x^m = b$ and $x^d = b$ have the same number of solutions in G.

(4) Let G be a group of order $p\cdot q$, where $p < q$ are primes. Let $x, y \in G$ of order p and q, respectively.

a) Argue that the cyclic subgroup $\langle y \rangle$ is normal in G.

b) Let $r \in \mathbb{N}$ be such that $xyx^{-1} = y^r$. Show that $r^p \equiv 1 \bmod q$.

(5) Let R be a commutative ring with identity. Suppose that for each $a \in R$ there exists some natural number $n \geq 2$ such that $a^n = a$ (note that n may depend on a). Show that every prime ideal of R is maximal.

(6) Let $A = K[X, Y]$ be the polynomial ring in two indeterminates over the field K, and consider the ideal I generated by $Y + X + X^3$ and $X^2 + 1$.

a) Show that $A/I \cong K[X]/(X^2 + 1)$.

Determine which of the indicated ideals is prime.

b) The ideal I if $K = \mathbb{Q}$.

c) The ideal I if $K = \mathbb{F}_p$, the finite field with p elements, where $p = 4m + 1$ for some integer $m \geq 1$.

d) The ideal $J = IS$ where $S = K[\![X, Y]\!]$, the power series ring in two variables over K.

(7) Denote by $\mathbb{F}_3$ the field with 3 elements. For each of the following polynomials, find the size of its splitting field. Support your answer.

a) $f = (X^2 + 1)(X^2 + 2X + 1) \in \mathbb{F}_3[X]$.

b) $g = (X^2 + 1)(X^3 + 2X + 2) \in \mathbb{F}_3[X]$.

c) $h = (X^2 + 1)(X^2 + 2X + 2) \in \mathbb{F}_3[X]$.

(8) a) Let $L \mid K$ be a field extension and let $r,\ s \in \mathbb{N}$ such that $\gcd(r, s) = 1$. Suppose $c \in L$ is such that $c^r \in K$ and $c^s \in K$. Show that $c \in K$.

b) Let K be a field of characteristic p. Suppose $a \in K$, but $a \notin K^p$ (that is, a is not the p-th power of an element in K). Show that the polynomial $X^p - a \in K[X]$ is irreducible.

[Hint: Work with a factorization of $X^p - a$ over its splitting field. How many distinct roots does the polynomial have?]

(9) a) State the primitive element theorem for field extensions.

b) Let α and β be roots of the polynomials $X^2 - 3$ and $X^2 - 5$ in $\mathbb{Q}[X]$, respectively, and let $L = \mathbb{Q}(\alpha, \beta)$. Explain why L is a simple field extension of $\mathbb{Q}$. Find an element $\gamma \in L$ such that $L = \mathbb{Q}(\gamma)$.

c) Let X, Y be indeterminates over $\mathbb{F}_2$, the finite field with 2 elements. Let $L = \mathbb{F}_2(X, Y)$ and $K = \mathbb{F}_2(u, v)$, where

$$u = X + X^2,\ v = Y + Y^2.$$

Explain why L is a simple extension of K. Find an element $\gamma \in L$ such that $L = K(\gamma)$.
[Hint: First show that X, Y, and $X + Y$ are all algebraic of degree 2 over K.]