June 2007

1. Let $C$ be a cyclic group of order 6. Find necessary and sufficient conditions on a group $G$ in order that $C \times G$ be a cyclic group.

2. Let $R$ be a non-zero ring containing $1 \neq 0$ such that the function $r \mapsto r^2$ from $R$ to $R$ is a homomorphism of rings. Prove that $R$ is a commutative ring of characteristic 2.

3. Prove that the group of automorphisms of the abelian group $\mathbb{Z}_3 \times \mathbb{Z}$ has 12 elements.

4. Precisely state the Sylow Theorems.
   Let $G$ be a group of order 105.
   Let $n_p(G)$ denote the number of $p$-Sylow subgroups of $G$ as usual.
   For each prime $p$ less than 10 determine possible values of $n_p(G)$ and the corresponding estimate of number of elements of order $p$ in $G$.
   Using these calculations or otherwise prove that $G$ cannot be simple.

5. Let $k \subset L$ be a Galois extension where $|Gal(L/k)| = 75$.
   (a) Prove that there is a unique field $F$ with $k \subset F \subset L$ such that $[F : k] = 3$.
   (b) Prove that the field $F$ constructed above is a Galois extension of $k$. 

Please turn over.
6. Let \( V \) be a 3-dimensional vector space over a field \( k \). Assume that you have three linear transformations \( f, g, h \) from \( V \) to \( k \) with the following properties.

- There is \( u \in V \) such that \( f(u) = 1, g(u) = h(u) = 0 \).
- There is \( v \in V \) such that \( g(v) = h(v) = 1 \).
- There is \( w \in V \) such that \( g(w) = 2, h(w) = 3 \).

Define a linear transformation from \( V \) to \( k^3 \) by \( L(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix} \).

Answer the following:

(a) Determine if \( L \) is surjective.
(b) Determine if \( L \) is injective.
(c) Determine \( \ker(f) \cap \ker(g) \cap \ker(h) \).

7. Consider the polynomial \( X^5 - 2 \) over \( \mathbb{Z}_{11} \) and let \( R = \mathbb{Z}_{11}[X]/(X^5 - 2) \).
As usual, identify \( \mathbb{Z}_{11} \) with its image in \( R \).
Define the ring homomorphism \( \sigma : R \to R \) by \( \sigma(t) = t^{11} \) for all \( t \in R \).

(a) Determine the order of \( \sigma \) (i.e. the smallest \( n \) such that \( \sigma^n = Id \)).
(b) Using the above or otherwise, argue that \( R \) is a field.
(c) Determine the Galois group of \( R \) over \( \mathbb{Z}_{11} \).

8. Give the precise definition of a prime ideal and a maximal ideal in a commutative ring \( T \) with \( 1 \neq 0 \).
Let \( R = \mathbb{Z}[X] \).

(a) Determine with proof if \( I = (X^4 + 2, X^2 + 1) \subset R \) is prime or maximal or neither.
(b) Determine with proof if \( J = (X^4 + X, X^2 + 1) \subset R \) is prime or maximal or neither.