Algebra Prelim

June 4, 2009

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

1. Let V be a vector space over a field F with basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and let a_1, a_2, a_3 be elements of F. Define a linear transformation on V by the rules $T(\mathbf{v}_i) = \mathbf{v}_{i+1}$ if $i < 4$ and $T(\mathbf{v}_4) = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$.

(a) Determine the matrix of T with respect to the given basis.

(b) Determine the characteristic polynomial of T.

2. In the vector space V of all polynomials $P(x) = c_0 + c_1x + c_2x^2 + c_3x^3$ of degree up to three and coefficients in $\mathbb{R}$, let W be the subset of all polynomials with

$$\int_0^1 P(x)\,dx = 0.$$

Verify that W is a subspace of V, determine the dimension of W and find a basis of W.

3. Let N be a normal subgroup of a group G with index $[G : N] = n$. Let $a \in G$ with $a^m \in N$ for some positive integer m. Assume that $\gcd(m, n) = 1$. Prove that $a \in N$.

4. Let R be a commutative ring. Suppose that every ideal I of R is prime. Prove that R is a field. (**Hint:** if $x \in R$, then $x \cdot x \in (x^2)$.)

5. Let R be a commutative ring. Let P be a prime ideal and let I and J be ideals of R. If $I \cap J \subset P$, prove that either $I \subset P$ or $J \subset P$.

6. (a) Find the unique (up to associates) factorization of 65 into a product of irreducibles in the ring of the Gaussian integers $\mathbb{Z}[i]$.

(b) Let $R = \mathbb{Q}[x]$. Let $f(x) = x^5 - 14x^3 - 98x + 7 \in R$ and assume that $f(x)$ divides the product $a(x)b(x)$ of two polynomials $a(x), b(x) \in R$. Prove that $f(x)$ divides either $a(x)$ or $b(x)$.

(c) Show that $Y^4 + 2x^2Y^3 - x$ is an irreducible polynomial in $\mathbb{Q}(x)[Y]$.

7. Let $f = x^3 - 3x + 1 \in \mathbb{Q}[x]$ and $u \in \mathbb{C}$ be a root of f.

(a) Show that f is the minimal polynomial of u over $\mathbb{Q}$.

(b) Write u^4 and u^6 as linear combination of 1, u, and u^2 with coefficients in $\mathbb{Q}$.

(c) Show that the element $w = 1 + u^2$ is nonzero and write w^{-1} as linear combination of 1, u, and u^2 with coefficients in $\mathbb{Q}$.

8. Let K/k be a field extension of characteristic $p \neq 0$, and let α be a root in K of an irreducible polynomial $f(x) = x^p - x - a$ over k.

(a) Prove that $\alpha + 1$ is also a root of $f(x)$.

(b) Prove that the Galois group of f over k is cyclic of order p.

9. Let $f = X^{12} - 1$.

(a) Compute the Galois group of f over the rational numbers.
Be sure to specify each element explicitly as an automorphism.

(b) Determine all subfields of the splitting field of f over the rational numbers.

10. Let $k \subset K$ be a Galois extension where $|\mathrm{Gal}(K/k)| = 45$.

(a) Prove that there is a *unique* field L with $k \subset L \subset K$ such that $[L : k] = 5$.

(b) Prove that the field L constructed above is a Galois extension of k.