Algebra Prelim
June 2, 2010

• Provide proofs for all statements, citing theorems that may be needed.
• If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.
• Do as many problems as you can and present your solutions as carefully as possible.

Good luck!

1. Let $V$ be the subspace of the $\mathbb{R}$-vector space of differentiable functions $\mathbb{R} \to \mathbb{R}$ that is generated by $f, \sin, \cos,$ and $\exp,$ where $f(x) = 1$ and $\exp(x) = e^x$ for all $x \in \mathbb{R}.$ Consider the linear map
   \[ \varphi : V \to V, \; g \mapsto g + g'. \]
   (a) Determine the presentation matrix of $\varphi$ with respect to the basis $\{f, \sin, \cos, \exp\}.$
   (You may use without proof that this is indeed a basis of $V.$)
   (b) Find all real eigenvalues of $\varphi.$
   (c) Determine a basis of the eigenspace of $\varphi$ to each real eigenvalue.

2. Let $A$ be an orthogonal matrix, that is, a square matrix with real entries such that $A \cdot A^T$ is the identity matrix. Show:
   (a) If $v$ and $w$ are eigenvectors of $A$ to the eigenvalues $1$ and $-1,$ respectively, then $v^T \cdot w = 0.$
   (b) The matrix $A$ is diagonalizable over the real numbers if and only if its minimal polynomial over $\mathbb{R}$ divides the polynomial $(X - 1)(X + 1).$

3. Let $H$ be a subgroup of a group $G,$ and let $g \in G$ be any element. Assume that the right coset $Hg$ equals some left coset of $H$ in $G.$ Show that then $gH = Hg.$

4. Let $a, b$ be elements of finite order in a group $G$ such that $ab = ba$ and $\langle a \rangle \cap \langle b \rangle = \{e\},$ where $e$ denotes the identity of $G.$ Show that the order of $ab$ is the positive least common multiple of the orders of $a$ and $b.$

5. Let $E/K$ be a Galois extension of degree 200. Prove that $E$ has a unique subfield $L$ containing $K$ such that $L/K$ is a Galois extension of degree 8.

6. Find a generator of the ideal $I$ of the polynomial ring $\mathbb{F}_5[X]$ that is generated by $X^4 + 4X^3 + 2X + 4$ and $X^3 + 3X^2 + 3$ (in other words, find a generator of the principal ideal $I$).
7. Let $K[X]$ be the polynomial ring over a field $K$, and consider

$$R := \{f \in K[X] \mid f'(0) = 0\},$$

where $f'$ denotes the derivative of $f$. Show:

(a) $R$ is a subring of $K[X]$, thus it is an integral domain.
(b) The elements $X^2$ and $X^3$ are irreducible in $R$.
(c) $R$ is not a factorial domain.

8. Let $\alpha := \sqrt[3]{5} + 1 \in \mathbb{R}$.

(a) Find the minimal polynomial $f$ of $\alpha$ over $\mathbb{Q}$.
(b) Argue that $\alpha$ is the only real root of $f$. (Hint: Consider $f(X + 1)$.)
(c) Determine the automorphism group of the field $\mathbb{Q}(\alpha)$.

9. Let $E \subset \mathbb{C}$ be the splitting field of $X^4 + 1$ over $\mathbb{Q}$. Determine explicitly all automorphisms and subfields of $E$.

10. Let $E/K$ be a Galois extension with Galois group $G$. Suppose there is an element $\alpha \in E$ such that $\sigma(\alpha) \neq \tau(\alpha)$ for all distinct automorphisms $\sigma, \tau \in G$. Prove that $E = K(\alpha)$. (Hint: Consider the Galois group of $E/K(\alpha)$.)