

Algebra Prelim

May 30, 2012

- Provide proofs for all statements, citing theorems that may be needed.
- If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.
- Do as many problems as you can and present your solutions as carefully as possible.

Good Luck!

- (1) Let K be a field and V be a finite-dimensional vector space over K . Let $F : V \rightarrow V$ be a diagonalizable linear map on V . Suppose that for all eigenvectors $v, w \in V$ of F the sum $v + w$ is either zero or also an eigenvector of F . Show that $F = \lambda \cdot \text{id}_V$.

[Hint: Show first that F has at most one eigenvalue.]

- (2) Let V be the vector space of all polynomial functions $p : \mathbb{R} \rightarrow \mathbb{R}$ such that $p(x) = c_0 + c_1x + c_2x^2$ for some $c_i \in \mathbb{R}$ (that is, all polynomial functions of degree at most 2). Define the following three linear functionals on V :

$$f_1(p) = \int_0^1 p(x)dx, \quad f_2(p) = \int_0^2 p(x)dx, \quad f_3(p) = \int_0^{-1} p(x)dx.$$

Show that $\{f_1, f_2, f_3\}$ is a basis of the dual space V^* .

- (3) Let $G = \langle x \rangle$ be a cyclic group of order n . For each integer $a \in \mathbb{Z}$ define the map

$$\sigma_a : G \longrightarrow G, \quad g \longmapsto g^a$$

Observe that σ_a is a group homomorphism (you need not prove this). Show the following:

- σ_a is an automorphism of G if and only if $(a, n) = 1$.
- Every group homomorphism $\varphi : G \rightarrow G$ is equal to σ_a for some a .
- $\sigma_a = \sigma_b$ if and only if $a \equiv b \pmod{n}$.

- (4) Let G be a group of order 30.
- Show that G has a normal subgroup of order 3 or a normal subgroup of order 5.
 - Show that if G has a normal subgroup of order 3 then it also has a normal subgroup of order 5.

[Hint: You may want to consider a quotient group.]

(please turn over)

- (5) Let $n \in \mathbb{N}_{\geq 2}$ and $m \in \mathbb{N}_{\geq 1}$. Show that the polynomial

$$\sum_{i=1}^n x_i^m - 1 \in \mathbb{Q}[x_1, \dots, x_n]$$

is irreducible.

[Hint: Induct on $n \geq 2$ and recall that $\mathbb{Q}[x_1, \dots, x_n] = \mathbb{Q}[x_1, \dots, x_{n-1}][x_n]$. There is hardly any computation involved in this problem.]

- (6) Let F be a field and $f \in F[x]$ be a polynomial of degree n . Suppose that f has n distinct roots in F . Show that the rings $F[x]/(f)$ and $\underbrace{F \times \dots \times F}_{n \text{ factors}}$ are isomorphic.
- (7) Let $f = (x^5 + 8x^3 + 18)(x^5 - 1) \in \mathbb{Q}[x]$. Let K be a splitting field of f and let $[K : \mathbb{Q}] = n$. Show that $20 \mid n$.
- (8) Let $f = x^9 - 1 \in \mathbb{Q}[x]$, and let K be a splitting field of f over \mathbb{Q} .
- Determine the number of subfields of K .
 - Determine the number of subfields of K that are Galois over \mathbb{Q} .
- (9) Let p be a prime number. Consider the finite fields \mathbb{F}_{p^4} and \mathbb{F}_p . We may assume that $\mathbb{F}_p \subseteq \mathbb{F}_{p^4}$.
- Show that there exist exactly $p^4 - p^2$ elements $a \in \mathbb{F}_{p^4}$ such that $\mathbb{F}_{p^4} = \mathbb{F}_p(a)$.
[Hint: Consider the subfields of \mathbb{F}_{p^4} .]
 - Show that there exist exactly $\frac{1}{4}(p^4 - p^2)$ monic irreducible polynomials of degree 4 in $\mathbb{F}_p[x]$.