Algebra Prelim
June, 2014

• Provide proofs for all statements, citing theorems that may be needed.
• If necessary you may use the results from other parts of this test, even though you may not have successfully proved them. Be sure to refer to such used parts.
• Do as many problems as you can and present your solutions as carefully as possible.

Good luck!

1. Consider a linear transformation $T$ on a vector space $V$ of dimension four over $\mathbb{R}$ the reals. On a basis $e_1, e_2, e_3, e_4$ of $V$, the transformation is defined by:

$$
T(e_1) = e_2, \quad T(e_2) = e_1, \quad T(e_3) = 2e_3 + e_4, \quad \text{and} \quad T(e_4) = e_3 - 2e_4.
$$

(a) Construct the matrix $A$ of the transformation in the given basis.
(b) Determine the characteristic polynomial, the eigenvalues and eigenspaces of $A$.
(c) Determine the kernel and the image of the transformation defined by the matrix $A^2 - I$ on $\mathbb{R}^4$.
(d) Is $A$ diagonalizable? Would you answer differently, if the field $\mathbb{R}$ is replaced by the field of rationals $\mathbb{Q}$?

2. Let $G$ be a finite group and $H$ a subgroup so that $[G : H] = d$ with $1 < d < |G|$.

(a) Briefly describe the natural homomorphism $\phi : G \to S_d$ where $S_d$ is considered to be the permutation group on the $d$ cosets of $H$ in $G$.
(b) Prove that if $|G|$ does not divide $d!$, then $H$ has a non-trivial subgroup $K$ such that $K \lhd G$.
(c) Using the above or otherwise show that a group $G$ of order 24 must contain a normal subgroup of order 4 or 8.

3. Let $\phi : \mathbb{Z}[X] \to \mathbb{Z}[X]/(X^2 + 7) = R$ be the natural residue class homomorphism. Let $\phi(X) = x$.

(a) Prove that $1 + x$ is irreducible in $R$.
(b) Prove that $(1 + x) \subset R$ is not a prime ideal.
(c) Is $R$ a U.F.D.? Why?
(d) Is $R$ a P.I.D.? If not then also present a concrete ideal of $R$ that is not principal.

4. Let $K$ be a field, $X, Y, t$ indeterminates and $\phi : K[X, Y] \to K[t]$ a $K$-algebra homomorphism. Let $P \subset K[X, Y]$ be the kernel of $\phi$.

(a) Prove that $P$ is a prime ideal of $K[X, Y]$.
(b) Assume that the image of $\phi$ is contained in $K$. Prove that $P$ is a maximal ideal of $K[X, Y]$.
(c) Let $\phi(X) = t^2 + 2, \phi(Y) = t^3 + 3$. Argue that $P$ is principal and find a generator for $P$. 

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5. Determine if the given polynomials are irreducible over the indicated fields.
   (a) \( f(X) = X^3 + X + 6 \) over \( \mathbb{Q} \).
   (b) \( g(Y) = Y^5 + X^2Y^4 - 3XY + X(1 + X) \) over \( \mathbb{Q}(X) \).

6. Let \( R \) be a commutative ring with \( 1 \neq 0 \). Recall that a proper ideal \( I \) of \( R \) is said to be primary if it satisfies the condition:

   \[
   \text{If } ab \in I \text{ then either } a \in I \text{ or } b \in \text{Rad}(I).
   \]

   The ideal \( \text{Rad}(I) \) is the set of elements \( x \) such that \( x^n \in I \) for some positive integer \( n \).

   (a) Briefly, explain why an ideal \( I \) is primary if and only if every zero divisor in the ring \( R/I \) is nilpotent.

   This form of the condition is often easier to check.

   (b) If \( Q \) is a primary ideal of \( R \), then prove that \( P = \text{Rad}(Q) \) is a prime ideal. We may express this by saying \( Q \) is \( P \)-primary.

   (c) Let \( A = K[X,Y] \), a polynomial ring in two variables over a field \( K \). Prove that \( I = (X+Y,Y^2) \) is primary in \( A \). Identify \( \text{Rad}(I) \).

7. Let \( K \) be a subfield of the reals and \( f(X) \) be a monic polynomial of degree \( n > 1 \) over \( K \).

   (a) Let \( r \) be the number of real roots of \( f(X) \) counted with multiplicity. Show that \( n - r \) is even.

   (b) Assume that \( K = \mathbb{Q} \) and that \( f(X) \) is irreducible of degree 3 with exactly one real root. Prove that the Galois group of the splitting field of \( f(X) \) over \( \mathbb{Q} \) is \( S_3 \).

   (c) Consider the cubic polynomial \( f(X) = X^3 - 3pX + 2p \), where \( p \) is a prime number of the form \( p = 1 + 3d^2 \) for some integer \( d \). Argue that the Galois group of the splitting field of \( f(X) \) over \( \mathbb{Q} \) is \( A_3 \). Where is the primeness of \( p \) used?

   You may use the formula that the discriminant of \( X^3 - 3aX + 2b \) is \( 108(a^3 - b^2) \).

8. Let \( f(X) = (X^3 - 5)(X^5 - 7) \in \mathbb{Q}[X] \), and let \( K \) be a splitting field of \( f(X) \) over \( \mathbb{Q} \). Let \( n = [K : \mathbb{Q}] \).

   (a) Argue that \( n \) is divisible by 15.

   (b) Show that \( K \) must contain a primitive 15-th root of unity over \( \mathbb{Q} \) which satisfies a monic polynomial of degree 8.

   (c) Deduce that \( n = 120 \).

9. Let \( f(x) = x^4 + x + 1 \in GF(2)[x] \) be a polynomial in \( x \) over the field \( GF(2) \) with two elements. Let \( K \) be a splitting field of \( f(x) \) over \( GF(2) \).

   (a) Determine \( [K : GF(2)] \).

   (b) Determine the Galois group of \( f(x) \) (i.e. \( Gal(K,GF(2)) \)).

   (c) Let \( \alpha \in K \) be a root of \( f(x) \). Give an explicit representation of all roots of \( f(x) \) in \( K \) in terms of \( \alpha \).

   (d) Determine the smallest number \( m \in \mathbb{N} \) such that \( f(x) \) divides \( (x^m - 1) \) in \( GF(2)[x] \).