Algebra Prelim, May 27, 2015

- Provide proofs for all statements, citing theorems that may be needed.
- If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.
- Do as many problems as you can and present your solutions as carefully as possible.

Good luck!

(1) Let $K$ be a field of characteristic not equal to 2. Let
\[ M := \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \in K^{2 \times 2} \]
and consider the linear map
\[ \varphi : K^{2 \times 2} \to K^{2 \times 2}, \quad X \mapsto MX - XM. \]

a) Find the matrix representation of $\varphi$ with respect to the standard basis of $K^{2 \times 2}$.

b) Find $\ker \varphi$.

[You may want to find first the kernel (null space) of the matrix representation from a). But your final answer to b) needs to be a subspace of $K^{2 \times 2}$]

c) Find all eigenvalues of $\varphi$.

d) Show that $\varphi$ is diagonalizable.

(2) Let $K$ be a field, $a$ an element in a field extension such that $a$ is algebraic over $K$. Denote by $f \in K[x]$ the minimal polynomial of $a$ over $K$. Consider the $K$-vector space $V := K[x]/(f)$ and the $K$-linear map
\[ \varphi : V \to V, \quad g + (f) \mapsto xg + (f). \]
Thus, $\varphi \in \text{End}(V)$. Show that the minimal polynomial of $\varphi$ is given by $f$.

(3) Let $G$ be a finite group and $\mathcal{X} = \{H \leq G\}$, that is, $\mathcal{X}$ is the set of all subgroups of $G$. Consider the action
\[ G \times \mathcal{X} \to \mathcal{X}, \quad (g, H) \mapsto gHg^{-1} \]
and denote by $\mathcal{O}_H$ the orbit of $H \in \mathcal{X}$. Show the following.

a) For any $H \in \mathcal{X}$ we have $|\mathcal{O}_H| = 1 \iff H \unlhd G$.

b) Let $p$ be a prime and $G$ be a nontrivial $p$-group. Let $n := |\mathcal{X}|$ and $m$ be the number of normal subgroups of $G$. Show that $p | (n - m)$.

(4) Consider the group $G := (\mathbb{Q}, +)/ (\mathbb{Z}, +)$.

a) Let $a, b \in \mathbb{Z}$ with $b \neq 0$ and suppose $\gcd(a, b) = 1$. Show that $\langle \frac{a}{b} + \mathbb{Z} \rangle = \langle \frac{1}{b} + \mathbb{Z} \rangle$ for the cyclic subgroups of $G$ generated by the given elements.

b) Show that for each $n \in \mathbb{N}$ there exists a unique subgroup of order $n$. 
5. Let $K$ be a field and $f, g \in K[x]$. Show that the following two statements are equivalent.
   a) There exists a ring homomorphism of the form
   \[ \varphi : K[x]/(f) \rightarrow K[x]/(g), \ p + (f) \mapsto p + (g). \]
   b) $g$ divides $f$ in $K[x]$.

6. Consider the ring $\mathbb{Z}[i]$ of Gaussian integers, and let $f$ be the ring homomorphism
   \[ f : \mathbb{Z} \rightarrow \mathbb{Z}[i]/(3 + 2i), \ c \mapsto c + (3 + 2i). \]
   Show the following.
   a) $f$ is surjective.
   b) $\ker f = 13\mathbb{Z}$.
   c) $|\mathbb{Z}[i]/(3 + 2i)| = 13$.
   [Hint: Have in mind that 2 and 3 are relatively prime.]

7. Let $[K : F] = n$ and let $a \in K$ such that there exist automorphisms $\sigma_1, \ldots, \sigma_n \in \text{Aut}(K \mid F)$ with $\sigma_i(a) \neq \sigma_j(a)$ whenever $i \neq j$. Show $K = F(a)$.

8. Consider the field extension $\mathbb{F}_{5^4} \mid \mathbb{F}_5$.
   a) Determine the number of elements $a \in \mathbb{F}_{5^4}$ satisfying $\mathbb{F}_{5^4} = \mathbb{F}_5(a)$.
   b) Determine the number of irreducible polynomials of degree 4 in $\mathbb{F}_5[x]$.

9. Denote by $\mathbb{Z}_n$ the cyclic group of order $n$.
   a) Find a field extension $K \mid \mathbb{Q}$ such that $\text{Gal}(K \mid \mathbb{Q}) \cong \mathbb{Z}_5$.
      [Hint: Start with a primitive 11th root of unity.]
   b) Let $L = K(\sqrt{2})$. Argue that $L \mid \mathbb{Q}$ is Galois and determine the cardinality of $\text{Gal}(L \mid \mathbb{Q})$.
   c) Give the isomorphism type of the Galois group $\text{Gal}(L \mid \mathbb{Q})$ and describe the automorphisms explicitly.