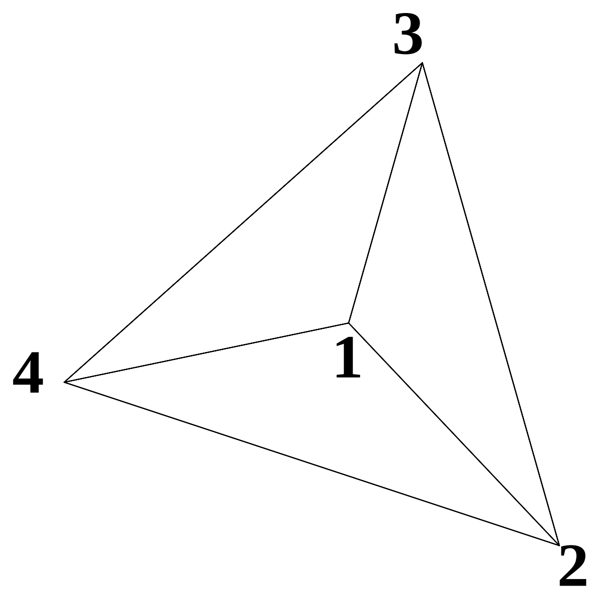
Algebra Prelim

Spring 1995

Provide proofs for all statements

1. Consider a regular tetrahedron with vertices labeled **{1, 2, 3, 4}**. Let G be the group generated by four rotations, one for each face that fixes the opposite vertex.



   (a) Represent the four generators as elements of S_4, the permutation group acting on the four vertices.

   (b) What is the orbit of the vertex **1**?

   (c) What is the order of the stabilizer in G of the vertex **1**?

   (d) What is the order of G?

   (e) This group should be well known to you. Write down various interesting facts about it, that you know. (Don't spend too much time on this until you are done with other problems.)

2. Let G be a group. Let $K \triangleleft G$ and $N \triangleleft G$, with $K \subseteq N$.

   (a) Describe the canonical homomorphism $\phi : G/K \longrightarrow G/N$. Be sure to prove that your homomorphism is well-defined.

(b) Calculate Ker ϕ and the image of ϕ.

(c) Prove that the factor group of G/K modulo Ker ϕ is a factor group of G.

3. Let $\mathbb{Q}$ denote the field of rational numbers and let $p(x) = x^4 + 9x + 3 \in \mathbf{Q}[x]$.

   (a) Show that the ring $F = \mathbb{Q}[x]/(p(x))$ is a field.

   (b) Find the dimension of F as a vector space over $\mathbb{Q}$ and list a basis for F over $\mathbb{Q}$.

   (c) $p(x)$ may also be regarded as an element in $\mathbb{Z}[x]$, where $\mathbb{Z}$ denotes the ring of integers. Is the ideal generated by $p(x)$ in $\mathbf{Z}[\mathbf{x}]$ a prime ideal? Is it a maximal ideal? (*Be sure to justify your answers.*)

4. Give an example of a ring R having an ideal I that **cannot be generated by** 2 elements. Can you do the same exercise when 2 is replaced by any fixed positive integer?

5. Show that if m and n are relatively prime positive integers, then the $\mathbb{Z}$-module $\mathbb{Z}/(m\mathbb{Z}) \bigoplus \mathbb{Z}/(n\mathbb{Z})$ is cyclic.

6. Let $\mathbf{F_2}$ denote the field with 2 elements.

   (a) Prove that there is a field K with 16 elements. Write down a polynomial $f(x) \in \mathbf{F_2}[x]$ such that K is a splitting field for $f(x)$ over $\mathbf{F_2}$.

   (b) Prove that there is no field having exactly 14 elements.

7. Define the terms *normal field extension* and *separable field extension.*

   Let K/F be a finite normal separable field extension with Galois group isomorphic to S_4.

   (a) What is $[K : F]$?

   (b) How many intermediate fields L are there for K/F such that $[L : F] = 3$?

   (c) What the total number of fields strictly between K and F?