MA 114 Worksheet # 10: Taylor Series & Taylor Polynomials

1. Find a power series representation for
   (a) $f(x) = x\cos(x^2)$.
   (b) $g(x) = (1+x)e^{-x}$.

2. Show that $\lim\limits_{x\to 0} \dfrac{e^x - \cos(x)}{\sin(x)} = 1$ using power series. Verify your answer with l'Hospital's Rule.
   [HINT: Write out the power series for each term and factor out the lowest power of x from the numerator and the denominator, and then consider the limit.]

3. What is $T_3(x)$ centered at $a = 3$ for a function $f(x)$ where $f(3) = 9$, $f'(3) = 8$, $f''(3) = 4$, and $f'''(3) = 12$?

4. Calculate the Taylor polynomials $T_2(x)$ and $T_3(x)$ centered at $x = a$ for the given function and value of a.
   (a) $f(x) = \tan x$, $a = \frac{\pi}{4}$
   (b) $f(x) = x^2 e^{-x}$, $a = 1$
   (c) $f(x) = \dfrac{\ln x}{x}$, $a = 1$

5. Let $T_2(x)$ be the Taylor polynomial of $f(x) = \sqrt{x}$ at $a = 4$. Apply the error bound to find the maximum possible value of $|f(1.1) - T_2(1.1)|$. Show that we can take $K = e^{1.1}$.

6. (a) Let $f(x) = 3x^3 + 2x^2 - x - 4$. Calculate $T_k(x)$ for $k = 1, 2, 3, 4, 5$ at both $a = 0$ and $a = 1$. Show that $T_3(x) = f(x)$ in both cases.
   (b) Let $T_n(x)$ be the n^{th} Taylor polynomial at $x = a$ for a polynomial $f(x)$ of degree n. Based on part (a), guess the value of $|f(x) - T_n(x)|$. Prove that your guess is correct using the error bound.