MA 114 Worksheet # 17: Integration by trig substitution

1. Conceptual Understanding:

   (a) Given the identity $\sin^2\theta + \cos^2\theta = 1$, prove that:
   $$\sec^2\theta = \tan^2\theta + 1.$$

   (b) Given $x = a\sin(\theta)$ with $a > 0$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, show that $\sqrt{a^2 - x^2} = a\cos\theta$.

   (c) Given $x = a\tan(\theta)$ with $a > 0$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, show that $\sqrt{a^2 + x^2} = a\sec\theta$.

   (d) Given $x = a\sec(\theta)$ with $a > 0$ and $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$, show that $\sqrt{x^2 - a^2} = a\tan\theta$.

2. Compute the following integrals:

   (a) $\displaystyle\int_0^2 \frac{u^3}{\sqrt{16 - u^2}}du$

   (b) $\displaystyle\int \frac{1}{x^2\sqrt{25 - x^2}}dx$

   (c) $\displaystyle\int \frac{x^3}{\sqrt{64 + x^2}}\,dx$

   (d) $\displaystyle\int_0^1 \sqrt{x^2 + 1}\,dx$

   (e) $\displaystyle\int \frac{x}{\sqrt{x^2 + 1}}\,dx$

3. Let $a, b > 0$. Prove that the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab.

4. Let $r > 0$. Consider the identity
   $$\int_0^s \sqrt{r^2 - x^2}\,dx = \frac{1}{2}r^2 \arcsin(s/r) + \frac{1}{2}s\sqrt{r^2 - s^2}$$
   where $0 \leq s \leq r$.

   (a) Plot the curves $y = \sqrt{r^2 - x^2}$, $x = s$, and $y = \dfrac{x}{s}\sqrt{r^2 - s^2}$.

   (b) Using part (a), verify the identity geometrically.

   (c) Verify the identity using trigonometric substitution.