MA 114 Worksheet # 26:
First-Order Linear Equations and Parametric Equations

1. Solve the second-order equation $xy'' + 2y' = 12x^2$ by making the substitution $u = y'$.

2. Consider a series circuit consisting of a resistor of R ohms, an inductor of L henries and a variable voltage source of $V(t)$ volts (time t in seconds). The current through the circuit $I(t)$ (in amperes) satisfies the differential equation
$$\frac{dI}{dt} + \frac{R}{L}I = \frac{1}{L}V(t).$$
Assume that $R = 110\ \Omega$, $L = 10$ H, and $V(t) = e^{-t}$ volts.

   (a) Solve the equation with initial condition $I(0) = 0$,

   (b) Calculate t_m and $I(t_m)$, where t_m is the time at which $I(t)$ has a maximum value.

3. A tank with a capacity of 400 liters is full of a mixture of water and chlorine with a concentration of 0.05 grams of chlorine per liter. In order to reduce the concentration of chlorine, fresh water is pumped into the tank at a rate of 4 liters per second. The mixture is kept stirred and is pumped out at a rate of 10 liters per second. Find the amount of chlorine in the tank as a function of time.

4. Conceptual Understanding:

   (a) How is a curve different from a parametrization of the curve?

   (b) Suppose a curve is parametrized by $(x(t), y(t))$ and that there is a time t_0 with $x'(t_0) = 0$, $x''(t_0) > 0$, and $y'(t_0) > 0$. What can you say about the curve near $(x(t_0), y(t_0))$?

5. Consider the curve parametrized by $c(t) = (\sin(t) + \frac{t}{\pi}, \left(\frac{t}{\pi}\right)^2)$, for $0 \leq t \leq 2\pi$.

   (a) Plot the points given by $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, 2\pi$.

   (b) Consider the derivatives of $x(t)$ and $y(t)$ when $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. What does this tell you about the curve near these points?

   (c) Use the above information to plot the curve.

6. Find a Cartesian equation for the following parametric curves. Sketch the curves to see if you solved them correctly.

   (a) $x = \sqrt{t}$, $y = 1 - t$.

   (b) $x = 3t - 5$, $y = 2t + 1$.

   (c) $x = \cos(t)$, $y = \sin(t)$.

7. Represent each of the following curves as parametric equations traced just once on the indicated interval.

   (a) $y = x^3$ from $x = 0$ to $x = 2$.

   (b) $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

8. A particle travels from the point $(2, 3)$ to $(-1, -1)$ along a straight line over the course of 5 seconds. Write down a set of parametric equations which describe the position of the particle for any time between 0 and 5 seconds.