Worksheet # 18: The Mean Value Theorem

1. State the mean value theorem and illustrate the theorem in a sketch.

2. (MA 113 Exam III, Problem 8(c), Spring 2009). Suppose that $g$ is differentiable for all $x$ and that $-5 \leq g'(x) \leq 2$ for all $x$. Assume also that $g(0) = 2$. Based on this information, is it possible that $g(2) = 8$?

3. Section 4.2 in the text contains the following important corollary which you should commit to memory:

   **Corollary 7, p. 284:** If $f'(x) = g'(x)$ for all $x$ in an interval $(a, b)$ then $f(x) = g(x) + c$ for some constant $c$.

   Use this result to answer the following questions:
   (a) If $f'(x) = \sin(x)$ and $f(0) = 15$ what is $f(x)$?
   (b) If $f'(x) = \sqrt{x}$ and $f(4) = 5$ what is $f(x)$?
   (c) If $f'(x) = k$ where $k$ is a constant, show that $f(x) = kx + d$ for some other constant $d$.

4. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers $c$ that satisfy the conclusion of the Mean Value Theorem.
   (a) $f(x) = e^{-2x}$, $[0,3]$  
   (b) $f(x) = \frac{x}{x + 2}$, $[1,4]$  

5. A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 miles on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

6. If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

7. For what values of $a$, $m$, and $b$ does the function
   \[
   f(x) = \begin{cases} 
   3 & \text{if } x = 0 \\
   -x^2 + 3x + a & \text{if } 0 < x < 1 \\
   mx + b & \text{if } 1 \leq x \leq 2 
   \end{cases}
   \]
   satisfy the hypotheses of the Mean Value Theorem on the interval $[0,2]$?

8. Determine whether the following statements are true or false. If the statement is false, provide a counterexample.
   (a) If $f$ is differentiable on the open interval $(a, b)$, $f(a) = 1$, and $f(b) = 1$, then $f'(c) = 0$ for some $c$ in $(a, b)$.
   (b) If $f$ is differentiable on the open interval $(a, b)$, continuous on the closed interval $[a, b]$, and $f'(x) \neq 0$ for all $x$ in $(a, b)$, then we have $f(a) \neq f(b)$.
   (c) Suppose $f$ is a continuous function on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$. If $f(a) = f(b)$, then $f'(\frac{a+b}{2}) = 0$.
   (d) If $f$ is differentiable everywhere and $f(-1) = f(1)$, then there is a number $c$ such that $|c| < 1$ and $f'(c) = 0$. 