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Gradient estimates for the Stokes semigroup subject Neumann boundary conditions in bounded convex domains

Abstract: On a bounded convex domain $\Omega \subset \mathbb{R}^d$, $d \geq 3$, we consider the instationary Stokes equations

$$
\begin{aligned}
\partial_t u(t, x) - \Delta u(t, x) + \nabla \pi(t, x) &= f(t, x) \quad (t > 0, x \in \Omega) \\
\text{div}(u(t, x)) &= 0 \quad (t > 0, x \in \Omega) \\
\partial_\nu u(t, x) - \nu \pi(t, x) &= 0 \quad (t > 0, x \in \partial \Omega) \\
u(t, x) &= u_0(x) \quad (x \in \Omega).
\end{aligned}
$$

These equations can be interpreted as an abstract evolution equation on $L^2(\sigma)(\Omega)$, the space of $L^2$-integrable solenoidal vector fields, where the solution operator is given by the semigroup $e^{-tA_2}$. Here $A_2$ denotes the Stokes operator on $L^2(\sigma)(\Omega)$. We prove that $A_2$ has a suitable realization $A_p$ on $L^p(\sigma)(\Omega)$, for $2 \leq p < \frac{2d}{d-1} + \varepsilon$, show that $-A_p$ generates an analytic semigroup on $L^p(\sigma)(\Omega)$ and that for every $\omega > 0$ there exists a constant $M_o > 0$ such that

$$
\|e^{-tA_p}u_0\|_{L^p} \leq M_o t^{-\frac{1}{2}} e^{\omega t} \|u_0\|_{L^p} \quad (u_0 \in L^p(\sigma)(\Omega))
$$

holds. This is done by a thorough investigation of the operator

$$
L^2(\Omega; \mathbb{C}^d) \ni f \mapsto |\omega + \lambda| (\omega + \lambda + A_2)^{-1} \mathbb{P} f
$$

$$
+ |\omega + \lambda|^\frac{1}{2} |\nabla (\omega + \lambda + A_2)^{-1} \mathbb{P} f| + |\omega + \lambda|^\frac{1}{2} |\pi + \Delta \mathbb{D}^{-1} \text{div}(f)|
$$

for $\lambda \notin \mathbb{R}_-$. Here $\Delta_D$ denotes the Dirichlet Laplacian on $W^{-1,2}(\Omega)$, $\pi$ the pressure 'belonging' to $(\omega + A_2)^{-1} \mathbb{P} f$ and $\mathbb{P}$ the Helmholtz projection. We prove that this defines a uniformly bounded family of sublinear operators on $L^2(\Omega; \mathbb{C}^d)$ for $\lambda$ in a sector $\Sigma_\theta := \{ z \in \mathbb{C} \setminus \{0\} : \arg(z) < \theta \}, \theta \in (0, \pi)$, and by prove weak reverse Hölder estimates for these operators.