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Gradient estiamtes for the Stokes semigroup subject Neumann boundary conditions in bounded convex domains

Abstract: On a bounded convex domain $\Omega \subset \mathbb{R}^d$, $d \geq 3$, we consider the instationary Stokes equations

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) + \nabla \pi(t,x) = f(t,x) & (t > 0, x \in \Omega) \\ \operatorname{div}(u(t,x)) = 0 & (t > 0, x \in \Omega) \\ \partial_\nu u(t,x) - \nu \pi(t,x) = 0 & (t > 0, x \in \partial\Omega) \\ u(0,x) = u_0(x) & (x \in \Omega). \end{cases}$$

These equations can be interpreted as an abstract evolution equation on $\mathrm{L}^2_\sigma(\Omega)$, the space of L^2-integrable solenoidal vector fields, where the solution operator is given by the semigroup e^{-tA_2}. Here A_2 denotes the Stokes operator on $\mathrm{L}^2_\sigma(\Omega)$. We prove that A_2 has a suitable realization A_p on $\mathrm{L}^p_\sigma(\Omega)$, for $2 \leq p < \frac{2d}{d-1} + \varepsilon$, show that $-A_p$ generates an analytic semigroup on $\mathrm{L}^p_\sigma(\Omega)$ and that for every $\omega > 0$ there exists a constant $M_\omega > 0$ such that

$$\|\nabla \mathrm{e}^{-tA_p} u_0\|_{\mathrm{L}^p_\sigma} \leq M_\omega t^{-\frac{1}{2}} \mathrm{e}^{\omega t} \|u_0\|_{\mathrm{L}^p_\sigma} \qquad (u_0 \in \mathrm{L}^p_\sigma(\Omega))$$

holds. This is done by a thorough investigation of the operator

$$\begin{aligned} \mathrm{L}^2(\Omega; \mathbb{C}^d) \ni f \mapsto & |\omega + \lambda| |(\omega + \lambda + A_2)^{-1} \mathbb{P} f| \\ & + |\omega + \lambda|^{\frac{1}{2}} |\nabla (\omega + \lambda + A_2)^{-1} \mathbb{P} f| + |\omega + \lambda|^{\frac{1}{2}} |\pi + \Delta_D^{-1} \operatorname{div}(f)| \end{aligned}$$

for $\lambda \notin \mathbb{R}_-$. Here Δ_D denotes the Dirichlet Laplacian on $\mathrm{W}^{-1,2}(\Omega)$, π the pressure 'belonging' to $(\lambda + A_2)^{-1} \mathbb{P} f$ and $\mathbb{P}$ the Helmholtz projection. We prove that this defines a uniformly bounded family of sublinear operators on $\mathrm{L}^2(\Omega; \mathbb{C}^d)$ for λ in a sector $\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : \arg(z) < \theta\}$, $\theta \in (0, \pi)$, and by prove weak reverse Hölder estimates for these operators.