Comments on Project 1 - Direction fields and Solution Curves

Some general comments on reading IODE project 1. *Experiment more*! Relate the ODE to the direction field and behavior of the solution curves!

(1) SCALES: Adjust the x and y scales as needed to capture all the behavior of the ODE. Many people did not do this for problems 3 and 4 and therefore did not see the full behavior of the solutions. For $f(x,y) = 2x(A-x)$, the x-axis must be large enough to contain 0 and A. Similarly, for the next ODE $f(x,y) = y(B-y)$, the y-axis must contain 0 and B.

(2) INITIAL CONDITIONS: Many people did not think about their initial conditions. You want to choose (x_0, y_0) so that a characteristic solution curve is show. For example, some people took $y_0 = 0$ for $y' = ky$. Of course the solution is $y(x) = C_0 e^{kx}$ so we must have $C_0 = 0$. This is not interesting!

(3) CHOICE OF STEP SIZE h: As the direction field becomes more complicated, one should vary h and take it small. This will smooth out the kinks many people found in solution curves, especially in ODEs 5 and 6.

(4) NUMBER OF LINE SEGMENTS: You can also change the number of line segments show for the direction field in the horizontal and vertical directions.

(5) RELATION TO THE EXACT SOLUTIONS: In the first four cases, the ODEs can be solved exactly. Not very many people did this. Why does the solution curve of $y' = 2x(A-x)$ look cubic? Because the solution is $y(x) = Ax^2 - (2/3)x^3 + C$. How about ODE 4? Is is similar to the logistic ODE we studied in section 2.5. Look there and see that the graphs are similar. The exact solution is given on page 82.

(6) PERIODICITY: Only a few people noted that the direction fields in ODEs 5 and 6 have periodicity in one direction. For ODE 5, $f(x,y) = x^2 + Ce^{x^2}\cos^2(3y)$. Notice that $f(x, y + (\pi k)/3) = f(x,y)$, for any integer k. How does this affect the plot of the direction field in the y-direction for fixed x (especially visible for x in $(-2,2)$)? Similarly, the direction field in ODE 6, $f(x,y) = -e^y - C(1-\cos(x))$ is periodic with period 2π in the x-direction. Take a look at your plots again to see this.

(7) ZEROS: The direction fields in 3 and 4 have zeros. What about ODEs 5 and 6? For 5, look at $f(x,y) = x^2 - e^{x^2}\cos^2(3y)$, where I took $C = -1$. The condition is $x^2 = e^{x^2}\cos^2(3y)$. This is complicated, but if $y = \pi/6 + (2\pi k)/3$, we have a zero when x is zero. So there are infinitely many zeros on the y-axis. Look at $f(x,y) = -e^y + (1-\cos(x))$ (I took $C = -1$.) When is it zero? This means $1 - \cos x = e^y$. Since $|\cos x| \leq 1$, the zeros can occur for $y = 0$ and $x = (2k+1)\pi$, for example. Do these zeros appear on the plots of the direction fields?

(8) EXPERIMENT: In ODE 6, $f(x,y) = -e^y - C(1-\cos(x))$, when y gets big, the term $-e^y$ dominated. What does this direction field look like? In ODE 5, when x is big, what does the ODE look like? The e^{x^2} term dominates the x^2 term.

1. Goal of Project I

Consider the ordinary differential equation

$$\frac{dy}{dx} = f(x;y). \tag{1.1}$$

We aim to understand graphically how properties of the function $f(x,y)$ determine the direction field. In particular, we consider functions f that are everywhere positive or negative, or that depend only on x or only on y. In Project 2 we will consider functions f that are periodic in x or in y.

2. Tips on using IODE

Use the menu item **Enter differential equation**, in the Direction Fields module of IODE, to enter a differential equation. Use the menu item **Change display parameters** to enter the domain and range of the plot, and then left-click on the graph to plot a solution curve. Make sure you use proper syntax when entering the ODE. See the on Matlab and IODE syntax.

3. Project Exercises

Let k and A, B, C be fixed real constants. Consider the following 6 ODEs, each with a different characteristic:

$$\frac{dy}{dx} = ky, \text{ where } k > 0 \tag{3.1}$$

$$\frac{dy}{dx} = ky, \text{where } k < 0 \tag{3.2}$$

$$\frac{dy}{dx} = 2x(A-x) \tag{3.3}$$

$$\frac{dy}{dx} = y(B-y) \tag{3.4}$$

$$\frac{dy}{dx} = x^2 + Ce^{x^2}\cos^2(3y) \tag{3.5}$$

$$\frac{dy}{dx} = -e^y - C(1-\cos(x)) \tag{3.6}$$

For each of the ODEs, complete the following three exercises:

(1) Choose two values for the constants in each ODE and make plots of the direction fields. Each plot should also show a solution curve with your choice of the initial condition. Use the **Enter caption** menu item to add your last name, problem number, and values of the constants and initial condition to the plot.

(2) For each direction field plotted in (1), describe in a few sentences what the distinctive features of the direction fields are. For example, for ODE (1) with $k = 2$ so $y' = 2y$, the distinctive feature is described as: *The direction field is horizontal at* $y = 0$.

(3) To identify distinctive features, it might help to blur your vision a little when looking at the direction field. Then check that your *distinctive feature* can be justified from the form of the ODE. For example, check that the ODE

$y' = 2y$ does imply that the slope of the tangent line to the solution curves is zero at $y = 0$.

Hand-in the two plots for each of the six ODEs plus a description of the direction field. PLEASE STAPLE your work.