Project 4 - Forced Oscillations

Due in class on Friday, 23 April 2010. Please staple your project!

These notes basically are from the IODE project and are a modification of those written by V. Zharnitsky and J. T. Tyson.

1. Second-order constant coefficient ODEs

Goal of the project is to consider how the solution of forced oscillator ODE depends on the various parameters. We consider ODEs of the form

$$mx''(t) + \gamma x'(t) + kx(t) = F(t), \tag{1.1}$$

where $\omega_0 = \sqrt{k/m}$ is the natural frequency of the oscillator, $\gamma \geq 0$ is the damping coefficient, and $F(t)$ is the external forcing term. We are going to investigate graphically the phenomena of: resonance, beating, transient behavior, and steady state oscillations. These all depend on the relation between ω_0, γ and the forcing term F.

2. Goals of the Project and Getting Started

First, start IODE and click on **Second order linear ODEs** in the main menu. You will get an interface very similar to the one for direction fields which we used in Projects 1 and 2. Go to the right side of the screen and click on **Solution Method**. It is normally set to Euler. Change it to Runge–Kutta. This is a more accurate numerical scheme. The Euler method is far too crude for what we are going to do in this project.

To ensure that you have the correct set up, try entering the following ODE:

$$x'' + x = 0. \tag{2.1}$$

Notice that when you enter an ODE into the **Enter differential equation** window, you must specify m, c, k and the forcing function $f(t)$ for the ODE:

$$mx''(t) + cx'(t) + kx(t) = f(t). \tag{2.2}$$

You must enter some number on each line even if it is zero. Do not leave empty spaces as IODE will not understand that. Now plot the solution with initial condition $(x(0), x'(0)) = (1, 0)$. Then plot the solution with initial condition $(x(0), x'(0)) = (0, 1)$. You should see on the screen the graphs of $\cos(t)$ and $\sin(t)$ respectively.

You may specify the initial conditions with the mouse and then IODE will plot the solution curves. Simply press down the left mouse button at the desired initial point $(t_0, x(t_0))$, and then drag the mouse at the desired slope a short distance to specify $x'(t_0)$, since you need to specify two conditions at t_0. When you release the mouse, IODE will plot the solution passing through this point with this slope. NOTE: If you just press down the left mouse button and release it without dragging, then the initial slope will be taken as one, that is $x'(t_0) = 1$.

3. The Project Components

3.1. **Beating solutions for an undamped, forced oscillator.** Set the display parameters to

$$-100 < t < 100, \text{ and } -30 < x < 30. \tag{3.1}$$

Plot the solution of the ODE:

$$x''(t) + x(t) = \cos(1.1t), \tag{3.2}$$

with initial condition $(x(0), x'(0)) = (0, 0)$. You should see a beautiful beat pattern!

(1) Graphically estimate the period of the beating (attach your solution plot), which is defined to be twice the distance between the neighboring troughs. Can you explain why it has this value?
Hint: The period of beating τ equals 2π divided by the frequency ω_b of beating: $\tau = 2\pi/\omega_b$. The trig identity
$$2\sin A \sin B = \cos(A - B) - \cos(A + B)$$
is useful to explain this phenomenon.

(2) Theoretical explanation: What happens to the period of beating as you change the forcing frequency from $\omega = 1.1$ to $\omega = 1.05$? Change the time scale to $-200 < t < 200$ in order to see the beats clearly. Attach a plot. What happens to the amplitude of the beating?

3.2. **Resonance behavior for an undamped, forced oscillator.** Reset the time scale to be $-100 < t < 100$. Plot the solution of the ODE:

$$x''(t) + x(t) = \cos t, \tag{3.3}$$

with initial condition $(x(0), x'(0)) = (0, 0)$. Note the forcing frequency $\omega = 1$ is equal to the natural frequency $\omega_0 = 1$ of the system. Make sure your scales are set so that you see an oscillating solution with growing amplitude. The solution should look like the graph of a function of the form $ct \sin t$. Graphically estimate the value of c using the slope of the envelope of the solution. Check your answer as follows.

(1) Use the **Plot arbitrary function** command to plot the amplitude of the envelope curve $x(t) = ct \sin t$. Do the two envelopes agree?
(2) Exactly solve the ODE and find the value of c.

Submit these results and plots.

Considering the graphs and solutions which you found in problem 1, in what sense (if any) can you say that the solution to the initial value problem

$$x''(t) + x(t) = \cos(\omega t), \; x(0) = 0; x'(0) = 0, \tag{3.4}$$

approaches the solution to the initial value problem at resonance $\omega_0 = \omega = 1$ considered here? **Hint:** Think about how the period and amplitude of the beating change as ω is decreased from 1.1 to 1.05 or as it is increased from 0.5 to 0.8 to 1.

3.3. **Behavior of a damped, unforced oscillator.** Change the scales to $0 < t < 50$ (you might adjust this to smaller t as γ gets bigger) and $-5 < x < 5$. We saw in class that the behavior of solutions to

$$mx''(t) + \gamma x'(t) + kx(t) = 0, \tag{3.5}$$

depend on the sign of the discriminant $\gamma^2 - 4mk$. Consider the ODE:

$$x''(t) + \gamma x'(t) + x(t) = 0, \tag{3.6}$$

and plot solutions to the initial value problem $(x(0), x'(0)) = (0, 2)$ for $\gamma = 0.1$ and for $\gamma = 0.5$. Describe the observed behavior in each case and relate it to the exact solutions.

3.4. **Transient behavior of a damped, forced oscillator.** The general ODE (1.1) has a general solution having the form

$$x(t) = x_h(t) + x_p(t), \tag{3.7}$$

where $x_h(t)$ is the most general solution of the associated homogeneous ODE (3.5) and $x_p(t)$ is a particular solution of the nonhomogeneous ODE.

We know that from class and from problem 3 that as long as $\gamma > 0$, we have that the amplitude goes to zero: $\lim_{t\to\infty} x_h(t) = 0$. So for large times, only the particular solution of (1.1) remains.

How large is large? When we do some plots described below, we see an initial behavior, called the *transient behavior*, this is when both $x_h(t)$ and $x_p(t)$ are contributing to the solution, followed by a regular pattern called the *steady-state behavior* that come mostly from x_p.

What is $x_p(t)$? Let's take $F(t) = F_0 \cos(\omega t)$. An intelligent guess is that it has the form

$$x_p(t) = A\cos(\omega t) + B\sin(\omega t). \tag{3.8}$$

We can determine A and B by substituting this x_p into the ODE. This is done in our text on pages 207-209 and discussed there. The result is that the steady-state solution looks like:

$$x_p(t) = \frac{F_0}{\Delta}\cos(\omega t - \delta), \tag{3.9}$$

where

$$\Delta = [m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]^{1/2}. \tag{3.10}$$

Note that when $\gamma \neq 0$, this never blows up, even at resonance $\omega = \omega_0$! But it is as big as it can be at resonance. The frequency of the steady-state solution is ω, the driving frequency.

For this example, use display parameters $0 < t < 30$ and $-5 < x < 5$. Plot solutions of the equation

$$x''(t) + x'(t) + x(t) = \cos(0.5t), \tag{3.11}$$

with the initial conditions $(x(0), x'(0)) = (0, 0), (0, 2), (0, 5)$. You should see that solutions with different initial conditions approach the same solution after some time τ. We say **transient** behavior occurs during the time interval $[0, \tau]$. After that time, all the different solutions appear to coincide on the graph and you observe only steady oscillations.

(1) Graphically estimate τ. Attach the plots.
(2) Can you explain why τ has this value? Give a theoretical explanation.

3.5. **Steady-State Behavior of a damped, forced oscillator at and near resonance.** Set the time scale at $0 < t < 50$ and the spatial scale at $-5 < x < 5$. We now consider a damped, forced oscillator at resonance:

$$x''(t) + 0.5x'(t) + x(t) = \cos(\omega t),\ x(0) = 0,\ x'(0) = 0, \tag{3.12}$$

Try varying the forcing frequency $\omega = 0.6, 1.0, 1.2$, below, at, and above the natural frequency ω_0. Attach the graphs.

(1) How does the amplitude $C(\omega)$ change with ω? When is $C(\omega)$ the largest?