MA113 Fall 2011
Some Useful Results: Exponentials and Logarithms
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Exponential Functions

For any base $a > 0$, the exponential function base a is defined by $f(x) = a^x$. If $a = 1$, then $f(x) = 1$ for all x, so this is not too interesting. The domain of a^x is all real numbers and the range is $(0, \infty)$. The exponential function is never zero for any finite x. The exponential function is continuous everywhere on its domain that is all real numbers. If the base $a > 1$, then $f(x) = a^x$ is an increasing function. If the base $0 < a < 1$, then the exponential function is a decreasing function.

For any $x, y \in R$, the laws of exponents are:

1. $a^x \cdot a^y = a^{x+y}$
2. $a^{xy} = (a^x)^y = (a^y)^x$
3. $a^{-x} = 1/a^x$

If both $a > 0$ and $b > 0$, then we have $ab > 0$ so ab is a base and $(ab)^x = a^x \cdot b^x$.

The most important base is $a = e > 1$. The number $e \sim 2.71828...$ is Euler's number.

Logarithmic Functions

The exponential function with base $a \neq 0$ is one-to-one on its domain. So, it has an inverse with domain $(0, \infty)$ and range R, all real numbers. The inverse function is the logarithm base a written $f(x) = \log_a x$. Remember that you can only put a positive number into a logarithm!

For a base $a > 0$ and for $x, y > 0$, the laws of logs are:

1. $c \log_a x = \log(x^c), \quad x > 0$
2. $\log_a(x/y) = \log_a x - \log_a y, \quad x, y > 0$
3. $\log_a(xy) = \log_a x + log_a y, \quad x, y > 0$

the log base e is called the natural log and written $\ln x$.

A very important formula relates a^x, for $a > 0$ to base e:

$$a^x = e^{x \ln a}, \quad a > 0$$

How do we change from one base to another? Recall that $\log_a x = y$ means that $a^y = x$. The easiest is to always change to base e. So taking the natural log of both sides of the last equation gives $y \ln a = \ln x$. Consequently, we get

$$\log_a x = \frac{\ln x}{\ln a}, \quad a > 0, \quad a \neq 1.$$

For two bases $a, b > 0$, with $a \neq 1$ and $b \neq 1$, we get

$$\log_a x = \left(\frac{\ln b}{\ln a}\right) \log_b x, \quad x > 0.$$