MA 114H — Calculus II Fall 2014
Sections 009–010

Exam 1 Sept. 23, 2014

Name: ____________________

Section: ____________________

Last 4 digits of student ID #: ________

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- **Multiple Choice Questions:** Record your answers on the right of this cover page by marking the box corresponding to the correct answer.
- **Free Response Questions:** Show all your work on the page of the problem. Clearly indicate your answer and the reasoning used to arrive at that answer.

Multiple Choice Answers

Question					
1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E

Exam Scores

Question	Score	Total
MC		20
5		16
6		16
7		16
8		16
9		16
Total		100

Unsupported answers for the free response questions might not receive credit!

Record the correct answer to the following problems on the front page of this exam.

1. (5 points) The power series of the function

$$f(x) = \frac{1}{1+7x} = \sum_{j=0}^{\infty} (-7x)^j = \sum_{j=0}^{\infty} (-1)^j 7^j x^j$$

about $c = 0$ is:

Note: $|7x| < 1$ so $|x| < \frac{1}{7}$ is need for convergence.

A. $\sum_{k=0}^{\infty} 7^k x^k$

B. $\sum_{k=1}^{\infty} 7^k x^k$

C. $\sum_{k=0}^{\infty} 7^k (x-1)^k$

D. $\sum_{k=0}^{\infty} (-1)^k 7^k x^k$

E. $\sum_{k=0}^{\infty} 7^k x^{k+1}$

2. (5 points) Let $C > 1$ be a fixed number. Which of the following answers is true for the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{Cn+17}$?

$a_n = \frac{(-1)^n n}{Cn+17}$

$\lim_{n\to\infty} |a_n| = 1/C \neq 0$

so the sequence diverges

A. The series is divergent.

B. The series is absolutely convergent.

C. The series is convergent, but not absolutely convergent.

D. The series is absolutely convergent, but not convergent.

E. None of the above.

Record the correct answer to the following problems on the front page of this exam.

3. (5 points) Which of the following ~~are~~ statements is true for a series $\sum_{n=1}^{\infty} a_n$? ~~Check all that apply.~~

A. If the series is convergent, then it is also absolutely convergent. False

B. If $\lim_{n\to\infty} a_n = 0$, then the series converges. False

C. If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. True

D. If the series is alternating, then it is convergent. False

E. None of the above. False

4. (5 points) Evaluate the series $\sum_{n=0}^{\infty} 2^{3-2n}$.

A. The series is divergent.

B. $\sum_{n=0}^{\infty} 2^{3-2n} = 6.$

C. $\sum_{n=0}^{\infty} 2^{3-2n} = 11.$

D. $\sum_{n=0}^{\infty} 2^{3-2n} = \frac{32}{3}.$

E. $\sum_{n=0}^{\infty} 2^{3-2n} = \frac{21}{2}.$

$2^{3-2n} = 2^3 \left(\frac{1}{2^2}\right)^n = 2^3 \left(\frac{1}{4}\right)^n$

So the series is geometric with ratio $1/4$

$\sum_{n=0}^{\infty} 2^{3-2n} = 2^3 \left(\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n\right)$

$= 2^3 \frac{1}{1-1/4} = 2^3 \cdot \frac{1}{3/4}$

$= \frac{8 \cdot 4}{3} = \frac{32}{3}.$

Free Response Questions: Show your work!

5. (16 points) Prove that the limit of the following sequence exists:

$$x_n = \frac{3n^2}{n^2+1}.$$

Clearly state the limit and present the proof using the ϵ definition.

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{3}{1+1/n^2} = 3 \quad \text{since } \lim_{n\to\infty} \frac{1}{n^2} = 0.$$

To prove this, given any $\varepsilon > 0$ we must find N_ε so $n > N_\varepsilon$ implies $|x_n - 3| < \varepsilon$.

Compute: $x_n - 3 = \dfrac{3n^2 - 3(n^2+1)}{n^2+1} = -\dfrac{3}{n^2+1}$

so $|x_n - 3| = \dfrac{3}{n^2+1} < \varepsilon$ is the desired condition.

Solve for n: $\dfrac{3-\varepsilon}{\varepsilon} < n^2$ or $n > \sqrt{\dfrac{3-\varepsilon}{\varepsilon}}$

We are only interested in $\varepsilon << 1$.

For M_ε take the smallest integer bigger than $\left[\dfrac{3-\varepsilon}{\varepsilon}\right]^{1/2}$.

6. (16 points) Use the Principle of Induction to prove the power law of differentiation for all positive powers $n \geq 1$:

$$\frac{d}{dx}x^n = nx^{n-1}, \quad n = 1, 2, 3, \ldots$$

Base Case: $n=1$ $\quad \frac{d}{dx}x = 1$ and $nx^{n-1} = 1$ when $n=1$.

Induction: Assume P_n: $\frac{d}{dx}x^n = nx^{n-1}$

P_{n+1} is a statement about $\frac{d}{dx}x^{n+1}$. Assume P_n and compute:

$$\frac{d}{dx}x^{n+1} = \frac{d}{dx}(x^n \cdot x) \overset{\text{product rule}}{=} \underbrace{\left(\frac{d}{dx}x^n\right)}_{\text{use } P_n} x + x^n$$

$$= nx^{n-1}x + x^n = (n+1)x^n$$

and this is precisely the statement P_{n+1}.

Free Response Questions: Show your work!

7. (16 points total) Determine whether the following series converges or diverges. Make sure to state all tests that you use.

(a) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

Positive terms

Ratio test $\dfrac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \dfrac{n!}{3^n n^2}$

$= 3\left(\dfrac{1}{n+1}\right)\left(\dfrac{n+1}{n}\right)^2$

The limit is 0

So this conv. absolutely.

(b) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{5+3^n}{100+4^n}$

$0 \leq a_n = \dfrac{5+3^n}{100+4^n} < \dfrac{5+3^n}{4^n}$ by comparison

Claim $\displaystyle\sum_{n=1}^{\infty} \frac{5}{4^n}$ & $\displaystyle\sum_{n=1}^{\infty} \left(\tfrac{3}{4}\right)^n$ both converge

so $\displaystyle\sum_n \frac{5+3^n}{4^n}$ converges. By the comparison test

this means $\sum a_n$ converges.

Now $\sum\left(\frac{1}{4}\right)^n$ & $\sum\left(\frac{3}{4}\right)^n$ are both geometric

with ratio less than 1 & so converge.

Alternate: $\dfrac{5+3^n}{100+4^n}$ compare to $\left(\dfrac{3}{4}\right)^n$ Use the Limit Comparison Test with $a_n = \dfrac{3^n[1+5/3^n]}{4^n[1+100/4^n]}$, $b_n = \left(\frac{3}{4}\right)^n$

Free Response Questions: Show your work!

8. (16 points) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ is absolutely convergent, conditionally convergent or divergent. Make sure to state all tests that you use.

The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ is an alt. series

+2

and the coefficients $\frac{1}{n^{1/3}} \to 0$ form

+4 a decreasing seq: $\frac{1}{n^{1/3}} > \frac{1}{(n+1)^{1/3}}$

8 By the alt. series test, this series

+2 converges. To check abs. conv.

Look at $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$. Apply the

12 +4 p-test. $\int_1^M \frac{dx}{x^{1/3}} = \frac{3}{2} x^{2/3} \Big|_1^M$

$= \frac{3}{2}(M^{2/3} - 1)$ and this has limit

$+\infty$ as $M \to \infty$. So the series

+4 $\sum \frac{1}{n^{1/3}}$ diverges. Hence $\sum \frac{(-1)^n}{n^{1/3}}$

16 converges conditionally.

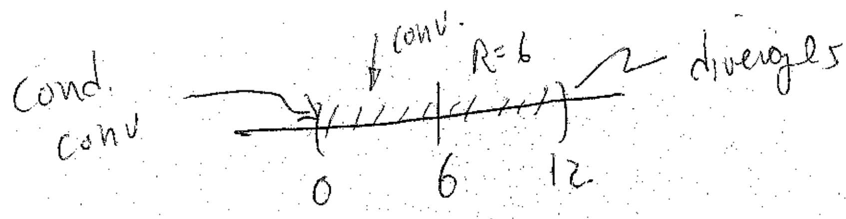
Free Response Questions: Show your work!

9. (16 points total) Consider the power series $\sum_{n=1}^{\infty} \frac{1}{n^{1/4}6^n}(x-6)^n$.

(a) (8 points) Find the radius of convergence.

Ratio test: $\frac{|x-6|^{n+1}}{(n+1)^{\frac{1}{4}} \cdot 6^{n+1}} \cdot \frac{n^{\frac{1}{4}} 6^n}{|x-6|^n} = \left(\frac{n}{n+1}\right)^{\frac{1}{4}} \frac{|x-6|}{6}$

Require: $\lim_{n \to \infty} \left(\frac{n}{n+1}\right)^{\frac{1}{4}} \frac{|x-6|}{6} < 1$ so $\boxed{R=6}$



(b) (8 points) Determine the behavior of the power series at the endpoints of the interval of convergence.

$\underline{x=12}$ $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{4}}}$ diverges by the p-test

$\underline{x=0}$ $\sum \frac{(-1)^n}{n^{\frac{1}{4}}}$ is conditionally conv. by the alt. series & p-tests.