MA 114H — Calculus II  Fall 2014
Sections 009–010

Exam 3  November 18, 2014

Name: $	extit{Solutions}$

Section: ____________________________

Last 4 digits of student ID #: _____

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- Multiple Choice Questions:
  Record your answers on the right of this cover page by marking the box corresponding to the correct answer.
- Free Response Questions:
  Show all your work on the page of the problem. Clearly indicate your answer and the reasoning used to arrive at that answer.

Multiple Choice Answers

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Exam Scores

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Unsupported answers for the free response questions might not receive credit!
1. (5 points) The region lying under the graph of \( f(x) = x^3 \) and over the interval \( x \in [0, 2] \) has centroid (that is, the center of mass coordinates):

A. \( \left( \frac{3}{5}, 2 \right) \)
B. \( \left( \frac{8}{5}, \frac{16}{7} \right) \)
C. (1, 4)
D. \( \left( \frac{8}{3}, \frac{16}{7} \right) \)
E. (1, \( \frac{16}{7} \))

\[
M = \int_0^2 x^3 \, dx = \left. \frac{x^4}{4} \right|_0^2 = \frac{16}{4} = 4
\]

You can also compute

\[
M_x = \int_0^2 y (2 - y^3) \, dy = 64 - 3 \cdot \frac{128}{7} = 64 - 3 \cdot \frac{128}{7}
\]

\[
M_y = \int_0^2 x^2 \, dx = \left. \frac{x^3}{3} \right|_0^2 = \frac{8}{3}
\]

2. (5 points) The surface area of the solid of revolution formed by rotation the graph of the function \( y(x) = 2x + 1 \) about the x-axis from \( x = 1 \) to \( x = 2 \) is:

A. \( 8\pi \sqrt{5} \)
B. \( \pi \sqrt{5} \)
C. \( 8\pi \)
D. \( 4\pi \sqrt{5} \)
E. \( 2\pi \)

\[
f'(x) = 2
\]

\[
S = 2\pi \int_1^2 f(x) \sqrt{1 + f'(x)^2} \, dx
\]

\[
= 2\pi \sqrt{5} \int_1^2 (2x+1) \, dx
\]

\[
= 2\pi \sqrt{5} \left( x^2 \bigg|_1^2 + x \bigg|_1^2 \right)
\]

\[
= 2\pi \sqrt{5} (3 + 1)
\]

\[
= 8\pi \sqrt{5}
\]
3. (5 points) The partial fraction decomposition of

\[ \frac{1}{(x^2 + 2)(x - 3)} = \frac{Ax + B}{x^2 + 2} + \frac{C}{x - 3} \]

is equal to which one of the following:

A. \( \frac{1}{11} \left( \frac{x^3 + 3}{x^2 + 2} + \frac{1}{x - 3} \right) \)

B. \( \frac{1}{11} \left( -\frac{x^3 + 3}{x^2 + 2} + \frac{1}{x - 3} \right) \)

C. \( \frac{1}{11} \left( \frac{x^3 - 3}{x^2 + 2} + \frac{1}{x - 3} \right) \)

D. \( \frac{1}{11} \left( \frac{x^3}{x^2 + 2} + \frac{1}{x - 3} \right) \)

E. \( \frac{1}{11} \left( \frac{x}{x^2 + 2} + \frac{1}{x - 3} \right) \)

\[ C = \frac{1}{11} \]
\[ A = -\frac{1}{11} \]
\[ B = -\frac{3}{11} \]

4. (5 points) The derivative of the function \( f(x) = \cosh(3x^2) \) is equal to:

A. \(-6x \sin(3x^2)\)

B. \(-6x \sinh(3x^2)\)

C. \(3x^2 \sinh(3x^2)\)

D. \(6x \sinh(3x^2)\)

E. \(6x \cosh(3x^2)\)

\[ f'(x) = \sinh(3x^2) \cdot 6x \]
\[ = 6x \sinh(3x^2) \]
5. (16 points) Evaluate the following indefinite integral:

\[ I = \int \frac{dx}{(x^2 + 1)^2} \]

You might need:

\[
\begin{align*}
\sin 2\theta &= 2 \sin \theta \cos \theta \\
\cos^2 \theta &= \frac{1}{2} (1 + \cos 2\theta) \\
\sin^2 \theta &= \frac{1}{2} (1 - \cos 2\theta)
\end{align*}
\]

\[
\begin{align*}
\tan \theta &= \frac{x}{1 + x^2} \\
\sec^2 \theta &= 1 + \tan^2 \theta
\end{align*}
\]

\[
\begin{align*}
(x^2 + 1)^2 &= \sec^4 \theta \\
I &= \int \frac{\sec^2 \theta}{\sec^4 \theta} \, d\theta = \int \cos^2 \theta \, d\theta \\
&= \frac{1}{2} \int d\theta + \frac{1}{2} \int \cos 2\theta \, d\theta = \frac{\theta}{2} + \frac{1}{4} \sin 2\theta + C
\end{align*}
\]

\[
\begin{align*}
\theta &= \tan^{-1} x \\
\sin 2\theta &= 2 \sin \theta \cos \theta = 2 \tan \theta \cos^2 \theta = \frac{2x}{\sec^2 \theta} = \frac{2x}{1 + x^2}
\end{align*}
\]

\[
\begin{align*}
\Rightarrow \quad I &= \frac{1}{2} \tan^{-1} x + \frac{1}{2} \frac{x}{1 + x^2} + C
\end{align*}
\]

Check: \[ I' = \frac{1}{2} \frac{1}{1 + x^2} + \frac{1}{2} \frac{1}{1 + x^2} - \frac{1}{2} \frac{x \cdot 2x}{(1 + x^2)^2} = \frac{1 + x^2}{(1 + x^2)^2} - \frac{x^2}{(1 + x^2)^2} = \frac{1}{(1 + x^2)^2} \]
6. (16 points) Compute the arc length of the curve \( f(x) = e^x \) on the interval \([0, a]\), for any \( a > 0 \). Use the substitution \( u = \sqrt{1 + e^{2x}} \), note that \( e^{2x} = u^2 - 1 \), and use partial fractions to do the \( u \) integral.

\[
L = \int_0^a \sqrt{1 + (f'(x))^2} \, dx = \int_0^a \sqrt{1 + e^{2x}} \, dx
\]

\[
u = \frac{1}{2} \left(1 + e^{2x}\right)^{\frac{1}{2}}
\]

\[
du = \frac{1}{2} \left(1 + e^{2x}\right)^{-\frac{1}{2}} 2e^x \, dx = \frac{e^{2x}}{\left(1 + e^{2x}\right)^{\frac{1}{2}}} \, dx = \frac{u^2 - 1}{u} \, dx
\]

so \( dx = \frac{u}{u^2 - 1} \, du \)

\[
L = \int \frac{u^2}{\sqrt{u^2 - 1}} \, du
\]

Indefinite integral:

\[
\int \frac{u^2}{u^2 - 1} \, du = \int \left(\frac{u^2 - 1}{u^2 - 1} + \frac{1}{u^2 - 1}\right) \, du
\]

\[
= u + \int \frac{du}{u^2 - 1}
\]

2nd indef. integral:

\[
\frac{1}{u^2 - 1} = \frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}
\]

\[
1 = A(u+1) + B(u-1) = (A+B)u + (A-B)
\]

\[
A = -B, \quad A - B = 1
\]

\[
A = \frac{1}{2}, \quad B = -\frac{1}{2}
\]

\[
\int \frac{du}{u^2 - 1} = \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| \quad \text{(in our definite integral \( u > \sqrt{2} - 1 \))}
\]

\[
L = \int \frac{u^2}{\sqrt{u^2 - 1}} \, du = \left\{ \left[1 + e^{2x}\right]^{\frac{1}{2}} \frac{u^2}{2} \right\} + \frac{1}{2} \left\{ \ln \left(1 + e^{2x} + 1\right) \right\} - \ln \left(\frac{u^2 - 1}{u^2 + 1}\right)
\]

L =
7. (16 points total) Differential equations.

(a.) (8 points) Find the most general solution:

\[
\frac{dy}{y - 3} = 2 \, dx \quad \ln |y - 3| = 2x + C \\
\begin{array}{l}
\text{Separate variables} \\
\hline
\end{array}
\]

\[
y'(x) = 2 \, y_0 \, e^{2x} = 2 \, (y - 3)
\]

(b.) (8 points) Find the unique positive solution for \( x \geq 0 \) to the initial-value problem:

\[
y(x) y'(x) = x^2 e^{-2y(x)} , \quad y(0) = 0,
\]

with \( x \geq 0 \) and \( y(x) \geq 0 \).

\[
yy' = x^2 e^{2y} \quad \text{Separate Variables} \quad (y e^{2y})' dy = x^2 dx
\]

\[
\begin{array}{l}
\int y e^{2y} \, dy = \int \frac{1}{4} e^2 \, dz = \frac{1}{4} e^{2y} \\
\hline
\end{array}
\]

\[
x = 2y \\
\frac{dz}{dy} = 4y \\ y = \frac{1}{4} e^{2y} = \frac{1}{3} x^3 + C \quad x > 0, \quad y > 0
\]

\[
so \quad y = \frac{1}{2} \ln \left( \frac{\frac{1}{3} x^3 + C}{\frac{1}{3} x^3 + C} \right) \quad \text{Check:} \quad y' = \frac{1}{2} \ln \left( \frac{\frac{1}{3} x^3 + C}{\frac{1}{3} x^3 + C} \right) \\
\]

\[
(\text{General:}) \quad y(x) = \left( \frac{1}{2} \ln \left( \frac{\frac{1}{3} x^3 + C}{\frac{1}{3} x^3 + C} \right) \right)^\frac{1}{2} \\
\]

\[
\text{Initial Condition:} \quad y(0) = 0 \quad \text{so} \quad \frac{1}{2} \ln C = 0 \quad \Rightarrow \quad C = 1 \quad y' = \left( \frac{x^2}{\frac{1}{3} x^3 + C} \right)^\frac{1}{2}
\]

\[
\text{Solution:} \quad y(x) = \left( \frac{1}{2} \ln \left( \frac{\frac{1}{3} x^3 + 1}{\frac{1}{3} x^3 + 1} \right) \right)^\frac{1}{2}
\]

\[
= x^2 e^{-2y^2}
\]
8. (16 points) Newton’s Law of Cooling states that the rate of change of the temperature $T(t)$ at time $t \geq 0$, measured in hours, of a body initially at temperature $T_1$ in an ambient environment at temperature $T_0 < T_1$ satisfies the initial value problem:

$$T'(t) = -k(T - T_0), \quad T(t = 0) = T_1, \quad k > 0.$$ 

How long does it take a cup of hot chocolate initially at temperature $T_1 = 80^\circ C$ in an room at $20^\circ C$ to cool to $50^\circ C$ if $k = \ln 2/\text{hour}$.

$$T_0 = 20 \quad T_1 = 80$$

$$T'(t) = -k(T - T_0) \quad \text{or} \quad \frac{dT}{T - T_0} = -k \, dt$$

$$\ln(T - T_0) = -kt$$

$$T = T_0 + Ae^{-kt}.$$ 

Initial Condition: $T(0) = 80 = 20 + A \Rightarrow A = 60.$

$$T(t) = 20 + 60e^{-kt} = 20 + 60e^{-(\ln 2)t}$$

At which time $t_1$ is $T(t_1) = 50$?

$$20 + 60e^{-(\ln 2)t_1} = 50$$

$$e^{-(\ln 2)t_1} = \frac{30}{60}$$

$$-(\ln 2)t_1 = \ln \frac{1}{2}$$

$$t_1 = 1 \text{ hour}$$
9. (16 points total) Let's consider the Simpson approximation method applied to the following integral.

(a.) (6 points) Write the integral for the arc length of the curve \( f(x) = \sin x \) with \( x \in [0, \pi] \). Do not evaluate the integral.

(b.) (6 points) Write Simpson's approximation for \( N = 4 \) for the integral found in part (a.) above.

(c.) (4 points) Evaluate Simpson's approximation of part (b.) exactly. This means leave quantities like \( \sqrt{2} \). Do not use a calculator.

\[
a) \quad f'(x) = \cos x \\
L = \int_0^\pi \sqrt{1 + \cos^2 x} \, dx \quad \text{let} \quad g(x) = \sqrt{1 + \cos^2 x} \\
\]

\[
b) \quad \text{Partition} \ [0, \pi] \ \text{into} \ 4 \ \text{subintervals} \quad \Delta x = \frac{\pi}{4} \quad 0 \quad \frac{\pi}{4} \quad \frac{\pi}{2} \quad \frac{3\pi}{4} \quad \pi \\
S = \frac{\Delta x}{3} \left[ g(0) + 4g\left(\frac{\pi}{4}\right) + 2g\left(\frac{\pi}{2}\right) + 4g\left(\frac{3\pi}{4}\right) + g(\pi) \right] \\
= \frac{\pi}{12} \left[ g(0) + 4g\left(\frac{\pi}{4}\right) + 2g\left(\frac{\pi}{2}\right) + 4g\left(\frac{3\pi}{4}\right) + g(\pi) \right] \\
\text{where} \quad g(x) = \sqrt{1 + \cos^2 x} \\
c) \quad \text{Calculate:} \quad g(0) = \sqrt{2} \quad g\left(\frac{\pi}{4}\right) = \sqrt{3/2} \quad g\left(\frac{\pi}{2}\right) = 1 \quad g\left(\frac{3\pi}{4}\right) = \sqrt{1/2} \quad g(\pi) = \sqrt{2} \\
\]

\[
S = \frac{\pi}{12} \left[ 2\sqrt{2} + 4\sqrt{3/2} + 2 \right] = \frac{\pi}{6} \left[ \sqrt{2} + 2\sqrt{3} + 1 \right].
\]