MA214-003 Spring 2009
Final Exam - 200 Points
8 May 2009, 1:00–3:00 PM, CB 339

INSTRUCTIONS: PLEASE WORK ALL 6 PROBLEMS BELOW. PLEASE WRITE YOUR NAME AND SECTION NUMBER ON EACH PAGE. NO CALCULATORS, BOOKS, PAPERS, OR NOTES ARE ALLOWED.

LAPLACE TRANSFORM FORMULAS ON THE LAST PAGE.

NAME: Solutions

PROBLEM	MAXIMUM GRADE	SCORE
1	30	30
2	35	35
3	35	35
4	35	35
5	35	35
6	30	30
TOTAL	200	200

1. (30 points). Find the unique solution to the first-order ODE:

$$y\frac{dy}{dx} = (1+x)(1+y^2), \quad y(0) = 0, \quad y \geq 0.$$

Separate variables: $\frac{y}{1+y^2}\,dy = (1+x)\,dx$

Integrate $\int \frac{y}{1+y^2}\,dy = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(1+y^2)$

$u = 1+y^2 \quad du = 2y\,dy$

$\int (1+x)\,dx = x + \frac{1}{2}x^2 + C$

so: $\frac{1}{2}\ln(1+y^2) = x + \frac{1}{2}x^2 + C$

$\ln(1+y^2) = 2x + x^2 + C$

$1+y^2 = Ce^{2x+x^2}$

$\boxed{y^2 = Ce^{2x+x^2} - 1}$

Initial Condition $y^2(0) = 0 = C - 1 \Rightarrow C = 1$

so $y = \pm\left[e^{2x+x^2} - 1\right]^{\frac{1}{2}}$

We require $y \geq 0$ so

$\boxed{y(x) = \left[e^{2x+x^2} - 1\right]^{\frac{1}{2}}}$

$y^2 = e^{2x+x^2} - 1$

$y^2 + 1 = e^{2x+x^2}$

Check $y' = \frac{1}{2}\left[e^{2x+x^2} - 1\right]^{-\frac{1}{2}} \cdot e^{2x+x^2} \cdot (2+2x)$

$yy' = (1+x)e^{2x+x^2} = (1+x)(1+y^2)$ ✓

$y(0) = 0$ & $y \geq 0$. ✓

2. (35 points). Find the unique solution to the second-order initial value problem:

$$y''(t) + 2y'(t) + 2y(t) = \delta(t-2), \quad y(0) = 1 \text{ and } y'(0) = 0.$$

Laplace transform:

$$(s^2+2s+2)(\mathcal{L}y)(s) = e^{-2s} + \underbrace{s+2}_{\text{from } y(0)=1}.$$

$$(\mathcal{L}y)(s) = \frac{e^{-2s}}{s^2+2s+2} + \frac{s}{s^2+2s+2} + \frac{2}{s^2+2s+2}$$

$s^2+2s+2 = (s+1)^2+1$ irred. quadratic

- $\dfrac{e^{-2s}}{(s+1)^2+1} = e^{-2s}H(s) \rightarrow u_2(t)(\mathcal{L}^{-1}H)(t-2) = u_2(t)e^{-(t-2)}\sin(t-2)$

  $(\mathcal{L}^{-1}H)(t) = e^{-t}\sin t$

- $\dfrac{s}{(s+1)^2+1} = \dfrac{s+1}{(s+1)^2+1} - \dfrac{1}{(s+1)^2+1}$ (remember: $(s-b)$ must be everywhere in the formula)

  $\downarrow$

  $e^{-t}\cos t - e^{-t}\sin t$

- $\dfrac{2}{(s+1)^2+1} \rightarrow 2e^{-t}\sin t$

Combine:

$$\boxed{y(t) = u_2(t)e^{-(t-2)}\sin(t-2) + e^{-t}\cos t + e^{-t}\sin t}$$

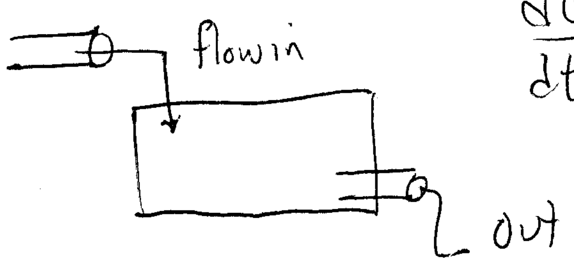
Check: $y(0) = 1$

$y'(t) = -e^{-t}(\cos t + \sin t) + e^{-t}(-\sin t + \cos t)$
$(0 \le t < 2)$

$y'(0) = -1 + 1 = 0.$

3. (35 points). A 100 liter tank initially contains 100 grams of salt. A solution of water and salt with a concentration of 0.5 grams per liter flows into the tank at a rate of 5 liters per minute. It is thoroughly mixed in the tank and flows out at the same rate. find the amount of salt in the tank as a function of time. How much salt is in the tank as time goes to infinity?

$Q(t) \equiv$ amount of salt in tank at time t in grams



$$\frac{dQ}{dt} = \left[\begin{matrix}\text{flow}\\ \text{in}\end{matrix}\right] - \left[\begin{matrix}\text{flow}\\ \text{out}\end{matrix}\right]$$

units: $\frac{\text{grams}}{\text{min}}$

flow in: $0.5 \frac{\text{grm}}{l} \cdot 5 \frac{l}{\text{min}} = 2.5\ g/\text{min}$

flow out: $\frac{Q(t)}{100} \frac{g}{l} \cdot 5 \frac{l}{m} = \frac{Q}{20}\ g/m$

$$\frac{dQ}{dt} = \left(2.5 - \frac{Q}{20}\right) \frac{g}{m}$$

$\frac{dQ}{dt} + \frac{1}{20} Q = 2.5$ solve: $\mu = e^{\frac{1}{20}t}$

$(\mu Q)' = 2.5\mu$ or $\mu Q = 50 e^{\frac{1}{20}t} + C$

$Q(t) = 50 + Ce^{-t/20}$

Initial Cond: $Q(0) = 100\,g$ $\quad 50 + C = 100 \quad C = 50$

$$\boxed{Q(t) = 50(1 + e^{-t/20})}$$

Equilibrium amount: $\boxed{\lim_{t \to \infty} Q(t) = 50 \text{ gms.}}$

4. (35 points). Consider the following nonhomogeneous, second-order ODE:

$$y''(t) + 2y'(t) + 5y(t) = e^{-t}.$$

a. Find a set of independent solutions for the associated homogeneous ODE. Make sure you verify that the two solutions are independent.

b. Find a particular solution to the nonhomogeneous ODE.

c. Write the general solution to the ODE.

a. homog ODE: $y'' + 2y' + 5 = 0$

$r^2 + 2r + 5 = 0$ roots: $\dfrac{-2 \pm [4-20]^{\frac{1}{2}}}{2} = -1 \pm 2i$

2 real solns: $y_1(t) = e^{-t}\cos 2t \quad y_2(t) = e^{-t}\sin 2t$

Wronskian: $y_1'(t) = -e^{-t}\cos 2t - 2e^{-t}\sin 2t \quad y_2'(t) = -e^{-t}\sin 2t + 2e^{-t}\cos 2t$

$W(t) = y_1 y_2'(t) - y_2 y_1'(t) = -e^{-2t}\cos 2t \sin 2t + 2e^{-2t}\cos^2 2t$
$- (-e^{-2t}\sin 2t \cos 2t - 2e^{-2t}\sin^2 2t) = 2e^{-2t} \neq 0$

so independent.

b. $y_p(t)$ <u>Method 1:</u> Guess $y_p(t) = Ae^{-t}$ $\boxed{y_p(t) = \frac{1}{4}e^{-t}}$

$y_p'(t) = -Ae^{-t}$

$y_p''(t) = Ae^{-t}$

Subst: $y_p'' + 2y_p' + 5y_p = (A - 2A + 5A)e^{-t} = e^{-t}$

$4A = 1 \quad A = \frac{1}{4}$

<u>Method 2:</u> $c_1' = -\dfrac{g y_2}{W} = -\dfrac{e^{-t} \cdot e^{-t}\sin 2t}{2e^{-2t}} = -\frac{1}{2}\sin 2t$

$c_1(t) = \frac{1}{4}\cos 2t$

$c_2' = \dfrac{g y_1}{W} = \dfrac{e^{-t}e^{-t}\cos 2t}{2e^{-2t}} = \frac{1}{2}\cos 2t \quad c_2(t) = \frac{1}{4}\sin 2t$

$y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t) = \frac{1}{4}\cos^2 2t \cdot e^{-t} + \frac{1}{4}\sin^2 2t \cdot e^{-t} = \frac{1}{4}e^{-t}.$

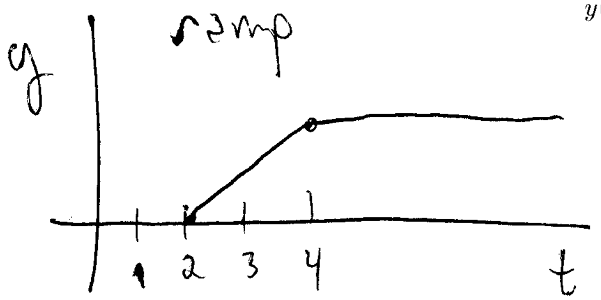
c. $\boxed{y(t) = C_1 e^{-t}\cos 2t + C_2 e^{-t}\sin 2t + \frac{1}{4}e^{-t}}$

5. (35 points). Find the unique solution to the nonhomogeneous, second-order ODE:

$$y''(t) + 4y(t) = \begin{cases} 0 & 0 \le t < 2 \\ (t-2)/2 & 2 \le t < 4 \\ 1 & t \ge 4, \end{cases}$$

with initial conditions:

$$y(0) = 0, \quad \text{and } y'(0) = 0.$$



$$g(t) = u_2(t)\left(\frac{t-2}{2}\right) - u_4(t)\left(\frac{t-2}{2}\right) + u_4(t)$$

$$= \left(\frac{t-2}{2}\right) u_2(t) - \frac{1}{2} u_4(t)(t-4)$$

LT: $(s^2+4)(\mathcal{L}y)(s) = \frac{1}{2}\frac{e^{-2s}}{s^2} - \frac{1}{2}\frac{e^{-4s}}{s^2}$

$$(\mathcal{L}y)(s) = \frac{1}{2}\frac{e^{-2s}}{s^2(s^2+4)} - \frac{1}{2}\frac{e^{-4s}}{s^2(s^2+4)}$$

partial fraction: $\frac{1}{s^2(s^2+4)} = \frac{A}{s^2} + \frac{Bs+C}{s^2+4}$ or $1 = As^2 + 4A + Bs^3 + Cs^2$

$B = 0$.

$A + C = 0$

$4A = 1$

$$= \frac{1}{4}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s^2+4}$$

$$\frac{1}{2}e^{-2s}\frac{1}{s^2(s^2+4)} = \frac{e^{-2s}}{8s^2} - \frac{1}{8}e^{-2s}\frac{1}{s^2+4} \rightarrow \frac{(t-2)}{8}u_2(t) - \frac{1}{16}\sin 2(t-2)\,u_2(t)$$

$$-\frac{1}{2}\frac{e^{-4s}}{s^2(s^2+4)} = -\frac{1}{8}\frac{e^{-4s}}{s^2} + \frac{1}{8}e^{-4s}\frac{1}{s^2+4} \rightarrow -\frac{(t-4)}{8}u_4(t) + \frac{1}{16}u_4(t)\sin 2(t-4)$$

Solution:

$$\boxed{y(t) = \frac{1}{8}\left[(t-2)u_2(t) - (t-4)u_4(t)\right] + \frac{1}{16}\left(u_4(t)\sin 2(t-4) - u_2(t)\sin 2(t-2)\right)}$$

6. (30 points). Consider the first-order ODE:

$$2\frac{dy}{dx} - y = e^{x/2}.$$

a. Find the most general solution to the ODE.

b. Find the unique solution to the ODE satisfying the initial condition: $y(0) = 0$.

a) $y' - \frac{1}{2}y = \frac{1}{2}e^{x/2}$ $\quad \mu = e^{-\frac{1}{2}x}$

$(\mu y)' = \mu q = \frac{1}{2}$ so $\mu y = \frac{1}{2}x + C$

$$\boxed{y(x) = \frac{1}{2}x e^{x/2} + Ce^{x/2}}$$

b) If $y(0) = 0$ then $0 = C$

$\Rightarrow$ $\boxed{y(x) = \frac{1}{2}x e^{x/2}}$

Check: $y' = \frac{1}{2}e^{x/2} + \frac{1}{4}x e^{x/2}$

$2y' = e^{x/2}\left(1 + \frac{1}{2}x\right) = e^{x/2} + \frac{1}{2}xe^{x/2}$

$= e^{x/2} + y.$ ✓

$y(0) = 0.$ ✓

Note: You may write the ODE as: $-(y + e^{x/2}) + 2y' = 0$.

$M = -(y + e^{x/2})$ $\quad N = 2$ $\quad$ This is not exact: $\frac{\partial M}{\partial y} = -1$ but $\frac{\partial N}{\partial x} = 0$

If you multiply by $e^{-x/2}$:

$$-(e^{-x/2}y + 1) + 2e^{-x/2}y' = 0$$

then $M = -(e^{-x/2}y + 1)$ and $\frac{\partial M}{\partial y} = -e^{-x/2}$

and $N = 2e^{-x/2}$ so $\frac{\partial N}{\partial x} = -e^{-x/2}$

so $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

and the ODE is now exact. This is what the integrating factor μ does. $\Psi = 2e^{-x/2}y - x = C$ so $y = \frac{x}{2}e^{x/2} + Ce^{x/2}$.