MA214-003 Fall 2008
Final Exam - 200 Points
19 December 2008, 1:00–3:00 PM, CB 239

INSTRUCTIONS: PLEASE WORK ALL 6 PROBLEMS BELOW. PLEASE WRITE YOUR NAME AND SECTION NUMBER ON EACH PAGE. NO CALCULATORS, BOOKS, PAPERS, OR NOTES ARE ALLOWED.

LAPLACE TRANSFORM FORMULAS ON THE LAST PAGE.

NAME: Solutions

PROBLEM	MAXIMUM GRADE	SCORE
1	35	
2	30	
3	35	
4	35	
5	35	
6	30	
TOTAL	200	

1. (35 points). Find the most general solution to the first-order ODE:

$$(x+y)+(x+y^2)\frac{dy}{dx}=0.$$

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Check $\quad \frac{\partial M}{\partial y} = 1 \quad \frac{\partial N}{\partial x} = 1$ so exact

let 10 $\begin{cases} 5 \quad \frac{\partial \psi}{\partial x} = M \quad \text{so} \quad \psi(x,y) = \frac{1}{2}x^2 + yx + h(y) \\ 5 \quad \frac{\partial \psi}{\partial y} = N \quad \text{so} \quad \psi(x,y) = xy + \frac{1}{3}y^3 + g(x) \end{cases}$

10 Comparing we take $g(x) = \frac{1}{2}x^2$ and $h(y) = \frac{1}{3}y^3$

5 so $\psi(x,y) = \frac{1}{2}x^2 + xy + \frac{1}{3}y^3$

The solution $y(x)$ satisfies $\psi(x, y(x)) = C$

10 or $\boxed{\frac{1}{2}x^2 + xy + \frac{1}{3}y^3 = C, \text{ any constant}}$

Check: $\quad x + y + xy' + y^2y' = 0$

$(x+y) + (x+y^2)y' = 0$ ✓

2. (35 points). Find the solution to the first-order initial value problem:

$$xy\frac{dy}{dx} = (1+y^2)^{1/2}, \quad x \geq 1,$$

and the condition $y(1) = 0$.

Separate variables:

5 $$\frac{y}{(1+y^2)^{\frac{1}{2}}}\,dy = \frac{1}{x}\,dx$$

Integrate: $u = 1+y^2$

5 $du = 2y\,dy$

$$\frac{1}{2}\int \frac{du}{u^{\frac{1}{2}}} = \int \frac{dx}{x} \quad 5$$

General soln: 5 $\boxed{(1+y^2)^{\frac{1}{2}} = \ln x + C, \quad (x \geq 1)}$

Initial Cond.: $y(1) = 0$

5 $$(1+y(1)^2)^{\frac{1}{2}} = \ln 1 + C$$

$$1 = C$$

5

Soln.: $\boxed{y(x)^2 = (\ln x + 1)^2 - 1}$ or $\boxed{y(x) = \pm\left[(\ln x+1)^2 - 1\right]^{\frac{1}{2}}}$

since $(\ln x+1)^2 - 1 \geq 0$.

Check: $2yy' = 2(\ln x+1)\frac{1}{x}$

$$xyy' = (1+y^2)^{\frac{1}{2}} \quad \checkmark$$

3. (35 points). Find the most general solution to the ODE:

$$xy'(x) + y(x) = \cos x, \quad x > 0.$$

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$$y'(x) + \frac{1}{x}y(x) = \frac{\cos x}{x}, \quad p(x) = \frac{1}{x}, \quad q(x) = \frac{\cos x}{x}$$

Integrating factor

$$\mu = e^{\int \frac{dx}{x}} = x \qquad 10$$

then

$$(\mu y)' = \mu q = \cos x \qquad 10$$

$$xy = \int \cos x \, dx = \sin x + C$$

$$\boxed{y(x) = \frac{\sin x}{x} + \frac{C}{x}, \quad x > 0} \qquad 10$$

Check:

$$y' = \frac{\cos x}{x} - \frac{\sin x}{x^2} - \frac{C}{x^2}$$

$$= \frac{\cos x}{x} - \frac{1}{x}y(x)$$

$$xy' + y = \cos x \quad \checkmark$$

4. (35 points). Consider the following nonhomogeneous, second-order ODE:

$$y''(t) - y'(t) - 6y(t) = 2e^{-2t}.$$

a. Find a set of independent solutions for the associated homogeneous ODE. Make sure you verify that the two solutions are independent.

b. Find a particular solution to the nonhomogeneous ODE.

c. Find the solution to the ODE

a) $r^2 - r - 6 = (r-3)(r+2) = 0$ 2 roots $r_1 = 3, r_2 = -2$

$y_1(t) = e^{3t}$ 5 $y_2(t) = e^{-2t}$ 5

$W(y_1, y_2)(t) = \begin{vmatrix} e^{3t} & e^{-2t} \\ 3e^{3t} & -2e^{-2t} \end{vmatrix} = -5e^{t} \neq 0$ so indep.

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b) Variation of Parameters (note: $g(t) = 2e^{-2t}$ is $2y_2$ so it is more involved to use undetermined coefficients - but you can)

$$u_1'(t) = -\frac{g y_2}{W}(t) = -\frac{2e^{-2t} \cdot e^{-2t}}{-5e^{t}} = \frac{2}{5} e^{-5t}$$

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$$u_1(t) = -\frac{2}{25} e^{-5t}$$

$$u_2'(t) = \left(\frac{g y_1}{W}\right)(t) = \frac{2e^{-2t} \cdot e^{3t}}{-5e^{t}} = -\frac{2}{5}$$

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$$u_2(t) = -\frac{2}{5} t$$

$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t) = -\frac{2}{15} - \frac{2}{5} t e^{-2t}$ 5

c) General Soln: $\boxed{y(t) = C_1 e^{3t} + C_2 e^{-2t} - \frac{2}{5} t e^{-2t}}$

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5. (35 points). Find the unique solution to the nonhomogeneous, second-order ODE:

$$y''(t) + 9y(t) = \delta(t-2) + u_2(t)e^{-(t-2)},$$

with initial conditions:

$$y(0) = 1, \quad \text{and } y'(0) = 0.$$

Laplace transform:

5 5

$$(s^2+9)(\mathcal{L}y)(s) - s = e^{-2s} + \frac{e^{-2s}}{s+1}$$

$$(\mathcal{L}y)(s) = \frac{e^{-2s}}{s^2+9} + \frac{e^{-2s}}{(s^2+9)(s+1)} + \frac{s}{s^2+9}$$

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$$= g_1(s) + e^{-2s}H(s) + g_2(s)$$

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$$g_1(s) = \frac{e^{-2s}}{s^2+9} = \frac{1}{3}e^{-2s}\left(\frac{3}{s^2+9}\right) \Rightarrow (\mathcal{L}^{-1}g_1)(t) = \frac{1}{3}u_2(t)\sin 3(t-2)$$

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$$g_2(s) = \frac{s}{s^2+3^2} \Rightarrow (\mathcal{L}^{-1}g_2)(t) = \cos 3t$$

$$H(s) = \frac{1}{(s^2+9)(s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+9} \quad \text{or} \quad (A+B)s^2 + (B+C)s + 9A + C = 1$$

$$\left.\begin{array}{l} A=-B \\ C=-B \end{array}\right\} A = C \qquad 10A = 1$$

$$A = 1/10 = C \qquad B = -1/10,$$

$$= \frac{1}{10}\frac{1}{s+1} - \frac{1}{10}\left[\frac{s}{s^2+9} - \frac{1}{s^2+9}\right]$$

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$$(\mathcal{L}^{-1}H)(t) = \frac{1}{10}e^{-t} - \frac{1}{10}\cos 3t + \frac{1}{30}\sin 3t$$

The term $e^{-2s}H(s)$ has ILT $u_2(t)\left[\frac{1}{10}e^{-(t-2)} - \frac{1}{10}\cos 3(t-2) + \frac{1}{30}\sin 3(t-2)\right]$

Finally, adding the terms;

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$$y(t) = \cos 3t + u_2(t)\left[\frac{11}{30}\sin 3(t-2) - \frac{1}{10}\cos 3(t-2) + \frac{1}{10}e^{-(t-2)}\right]$$

6. (30 points). The population of bacteria $P(t)$ satisfies the ODE:

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P,$$

for $t \geq 0$. The constants $r > 0$ and $K > 0$ represent the growth rate and the saturation level, respectively.

a. Find the population if the initial population $P(0)$ satisfies $0 < P(0) < K$.
   HINT: use partial fractions to integrate.

b. What is $\lim_{t\to\infty} P(t)$ in case (a.)?

a) Separate variables

$$\frac{dP}{P(k-P)} = \frac{r}{k}\,dt \qquad 5$$

Integrate: $\dfrac{1}{P(k-P)} = \dfrac{A}{P} + \dfrac{B}{k-P} \Rightarrow 1 = kA - PA + BP$

so $B - A = 0$, $A = 1/k$, $B = \frac{1}{k}$

$$\frac{1}{k}\int \frac{dP}{P} + \frac{1}{k}\int \frac{dP}{k-P} = \frac{r}{k}t + C \qquad 5$$

$$\frac{1}{k}\ln P - \frac{1}{k}\ln(k-P) = \frac{r}{k}t + C$$

$$\ln\left(\frac{P}{k-P}\right) = rt + C \quad \text{or}^{5} \quad \boxed{\frac{P}{k-P} = C_0 e^{rt}} \quad \text{general soln.}$$

$t=0$ $\quad \dfrac{P(0)}{k-P(0)} = C_0$ $\quad 5$ $\quad$ note $k - P(0) > 0$ by the stated condition.

so $\quad P(t) = (k - P(t))e^{+rt}\left(\dfrac{P_0}{k-P_0}\right) \quad P_0 = P(0)$

Solve for $P(t)$:

$$5 \qquad P(t)\left[1 + \frac{P_0}{k-P_0}e^{rt}\right] = \frac{kP_0}{k-P_0}e^{+rt}$$

$$\boxed{P(t) = \left(\frac{kP_0}{k-P_0}\right)\frac{e^{rt}}{1 + \left(\frac{P_0}{k-P_0}\right)e^{rt}}}$$

check: $P(0) = \dfrac{kP_0}{k-P_0}\Big/\left(\dfrac{1}{1+\frac{P_0}{k-P_0}}\right) = P_0$

b) $$\lim_{t\to\infty} P(t) = \frac{kP_0}{k-P_0}\cdot\frac{\cancel{e^{rt}}}{\cancel{e^{rt}}}\left(\frac{1}{e^{-rt} + \frac{P_0}{k-P_0}}\right) = k, \text{ the saturation level}$$

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