

**FINAL EXAM: 19 December 2008, 1:00–3:00, CB 239**

1. Find the most general solution to the ODE:

$$(2xe^y + e^x) + (x^2 + 1)e^y \frac{dy}{dx} = 0.$$

2. Find the most general solution to the ODE:

$$x \frac{dy}{dx} + y = \sin x.$$

3. Find the most general solution to the ODE:

$$3 \frac{dy}{dx} - y = xe^{x/3}.$$

What is the unique solution satisfying  $y(0) = 2$ ?

4. Find the most general solution to the ODE:

$$xe^y \frac{dy}{dx} + \frac{x^2 + 1}{y} = 0.$$

5. A tank contains 200 l of a dye solution with a concentration of 1 g/l. The tank is rinsed with fresh water flowing in at a rate of 2 l/min. A well-stirred solution flows out at the same rate. How long does it take for the dye concentration in the tank to reach 1% of its original value?

6. Find the unique solution to the ODE:

$$y' + y = 5 \sin 2t,$$

with initial condition  $y(0) = 1$ .

7. Find the unique solution to the ODE:

$$2y' + y = 3t^2,$$

with  $y(1) = 0$ .

8. The population of a certain mammal  $P(t)$  satisfies the ODE:

$$\frac{dP}{dt} = r \left( 1 - \frac{P}{K} \right) P,$$

for  $t \geq 0$ . The constants  $r > 0$  and  $K > 0$  represent the growth rate and the saturation level, respectively. Find the population  $P(t)$  if the initial population  $P(0)$  satisfies  $0 < P(0) < K$ , and if it satisfies  $P(0) > K$ .

9. Find a general formula for the Wronskian of two solutions to the ODE:

$$2t^2 y'' + 4ty' - y = 0.$$

When can we be sure that the two solutions are independent?

10. Find the unique solution to the initial-value problem:

$$y'' + 4y' + 5y = 0,$$

with  $y(0) = 1$ , and  $y'(0) = 0$ .

11. Consider the following nonhomogeneous, second-order ODE:

$$y''(t) - y'(t) - 6y(t) = 2e^{3t}.$$

Find a set of independent solutions to the associated homogeneous ODE.  
Find the unique solution to the nonhomogeneous ODE with  $y(0) = 0$  and  $y'(0) = 1$ .

12. Find a set of independent solutions for the ODE:

$$y'' - 6y' + 9y = 0.$$

13. Find a particular solution to the ODE:

$$4y'' - 4y' + y = 16e^{t/2}.$$

14. Find the unique solution to the first-order initial-value problem:

$$\sqrt{2xy} \frac{dy}{dx} = 1, \quad x > 0,$$

and the condition  $y(0) = 1$ .

15. a. Sketch the function  $h(t)$  and write it, for  $t \geq 0$ , in terms of the step functions  $u_c(t)$ :

$$h(t) = \begin{cases} 0 & 0 \leq t < 1 \\ t - 1 & 1 \leq t < 2 \\ 1 & t \geq 2. \end{cases}$$

- b. Find the Laplace transform of the function in (a.).  
c. Find the inverse Laplace transform of the function

$$F(s) = \frac{e^{-4s} + e^{-2s}}{s^2 + 2s + 1}.$$

16. Find the unique solution to the ODE:

$$y'' + y = 0,$$

with  $y(\pi/3) = 2$ , and  $y'(\pi/3) = -4$ .

17. Find the unique solution to the initial value problem:

$$y''(t) + 2y(t) = \begin{cases} \cos t & 0 \leq t < \pi \\ 0 & t \geq \pi, \end{cases}$$

with  $y(0) = 0 = y'(0)$ . Note that  $\cos(t - \pi) = -\cos t$ .

18. Find the inverse Laplace transform of the function

$$F(s) = \frac{e^{-2s}}{s^2 + 2s + 3} + 2e^{-4s}.$$