Final Review - 1

1. Check if an exact ODE: $M(x,y) = 2xe^y + e^x \qquad N(x,y) = (x^2+1)e^y$

$$\partial_y M = 2xe^y \stackrel{?}{=} \partial_x N = 2xe^y \quad \checkmark$$

Find $\psi(x,y)$ so $\partial_x \psi = M \Rightarrow \cancel{M}(x,y) = x^2e^y + e^x + h(y)$

$$\partial_y \psi = N \Rightarrow \psi(x,y) = x^2e^y + e^y + g(x)$$

Compare: $h(y) = e^y \quad g(x) = e^x$ so $\psi(x,y) = x^2e^y + e^x + e^y$

Soln $y(x)$ satisfies $\psi(x, y(x)) = C$, a constant.

2. General form: $y' + py = q$ with $p = \frac{1}{x}$ and $q = \frac{\sin x}{x}$

Integrate for $\mu = e^{\int \frac{1}{x}dx} = x$

$$(\mu y)' = (xy)' = \mu q = \sin x \Rightarrow xy = -\cos x + C$$

$$y(x) = -\frac{\cos x}{x} + \frac{C}{x}.$$

3. Same idea: $y' - \frac{1}{3}y = \frac{1}{3}xe^{x/3} \quad p(x) = -\frac{1}{3} \quad \mu = e^{-\frac{1}{3}x}$

$$(e^{-1/3x}y)' = \mu q = \tfrac{1}{3}x e^{-\frac{1}{3}x} \cdot e^{\frac{1}{3}x} = \tfrac{1}{3}x$$

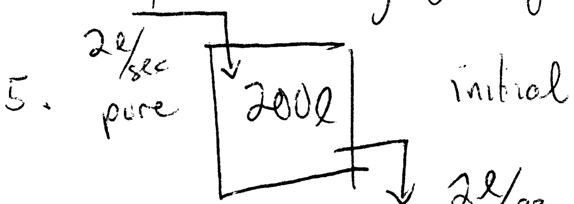
Integrate: $e^{-x/3}y = \frac{1}{6}x^2 + C \qquad y(x) = \frac{1}{6}x^2e^{x/3} + Ce^{x/3}$

If $y(0) = 2$ then $2 = C$.

4. Separate variables: $xe^y y' = -\frac{(x^2+1)}{y}$ so $ye^y dy = -(x + \frac{1}{x})dx$

Integrate $\int ye^y dy = ye^y - e^y = -\frac{1}{2}x^2 - \ln|x| + C$

$$e^y(y-1) = -\frac{1}{2}x^2 - \ln|x| + C.$$

5. 

initial concentration: $1\, g/\ell$

$Q(t) \equiv$ amount of dye in grams in tank at time t.

$$Q(0) = 1\, g/\ell \cdot 200\,\ell = 200g.$$

$$\frac{dQ}{dt}\left[\frac{g}{sec}\right] = \begin{bmatrix} \text{flow in} \\ \| \\ 0 \end{bmatrix} - \left[\frac{Q(t)}{200}\frac{g}{\ell} \cdot 2\frac{\ell}{sec}\right] = -\frac{Q(t)}{100}$$

pure H_2O.

$$\frac{dQ}{dt} = -\frac{Q(t)}{100}\, g/sec \qquad \frac{dQ}{Q} = -\frac{dt}{100} \qquad \ln Q = -t/100 + C$$

$$Q(t) = Q_0 e^{-t/100} = 200e^{-t/100}$$

Find t_0 s.t. $Q(t_0) = 2 = 200e^{-t/100}$ Conc. $0.01\, g/\ell \times 200\,\ell = 2g$

$$10^{-2} = e^{-t_0 10^{-2}} \qquad \log 10^{-2} = -10^{-2}t_0 \qquad t_0 = 100 \log 100.$$

Final Review - 2

6. $y' + y = 5\sin 2t \qquad p = 1 \qquad \mu = e^{\int p} = e^{t}$

$(e^{t}y)' = 5e^{t}\sin 2t$ integrate by parts.

7. $y' + \frac{1}{2}y = \frac{3}{2}t^{2} \qquad p(t) = \frac{1}{2} \qquad \mu = e^{t/2}$

$(e^{t/2}y)' = \frac{3}{2}e^{t/2}t^{2}$ Integrate by parts.

8. $\frac{dP}{dt} = \frac{r}{k}(k-P)P$ Separate: $\frac{dP}{P(k-P)} = dt\left(\frac{r}{k}\right)$

Integrate: $\frac{A}{P} + \frac{B}{k-P} = \frac{1}{P(k-P)} \Rightarrow Ak - AP + BP = 1$

$-A + B = 0 \quad A = 1/k \quad B = 1/k$

$\frac{1}{k}\ln P - \frac{1}{k}\ln k-P = \frac{r}{k}t + C$ Solve.

9. Abel's formula - need $y'' + py' + qy = 0$

$W(y_1, y_2)(t) = C_0\exp\{-\int p(t)\,dt\}$

so $y'' + \left(\frac{2}{t}\right)y' + \frac{1}{2t^2}y = 0 \qquad p(t) = \frac{2}{t} \qquad \int p(t)dt = 2\ln t$

$W(t) = C_0 e^{-2\ln t} = C_0/t^2, \; t > 0.$

10. $y'' + 4y' + 5y = 0 \qquad r^2 + 4r + 5 = 0$

$r_1, r_2 = \frac{-4 \pm [16-20]^{1/2}}{2} = -2 \pm i \qquad e^{(-2\pm i)t} = e^{-2t}(\cos t \pm i\sin t)$

2 real solns: $y_1(t) = e^{-2t}\cos t \qquad y_2(t) = e^{-2t}\sin t$

General Soln: $y(t) = C_1e^{-2t}\cos t + C_2e^{-2t}\sin t$

Solve $y(0) = 1 \quad y'(0) = 0$ for C_1 & C_2.

11. $y'' - y' - 6y = 2e^{3t}$ Homog ODE: $r^2 - r - 6 = (r-3)(r+2) = 0$

2 solns $y_1(t) = e^{3t} \qquad y_2(t) = e^{-2t} \qquad W(t) = \begin{vmatrix} e^{3t} & e^{-2t} \\ 3e^{3t} & -2e^{-2t}\end{vmatrix} = -5e^{t}$

Since the nonhomog term $2e^{3t}$ is a multiple of y_1, one can try $y_p(t) = Ate^{3t}$ or use Variation of parameters

$y_p'(t) = Ae^{3t} + 3Ate^{3t} = A(1+3t)e^{3t} \qquad y_p''(t) = 3A(1+3t)e^{3t} + 3Ae^{3t}$

Subst. $6A + \cancel{9At} - A - \cancel{3At} - \cancel{6At} = 2 \qquad A = 2/5 \qquad y_p(t) = 2/5\,te^{3t}$

General soln: $y(t) = C_1e^{3t} + C_2te^{-2t} + 2/5\,te^{3t}$