MA214–03 Fall 2009
Quiz #5 - 10 Points
13 November 2009

NAME: Solution

1. (10 points). Find the motion of an undamped harmonic oscillator with mass $m = 1kg$ (kilogram) and spring constant $k = 4kg/sec^2$ (kilograms per second per second) driven by a cosine function at resonance with amplitude $2kg/sec^2$. The oscillator is initially at rest.

You might need:

$$\cos^2 A = \frac{1}{2}(1 + \cos(2A))$$

$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$

Amplitude

Write the ODE: $\omega_0 = \sqrt{\frac{k}{m}} = 2$ $\quad g(t) = 2\cos 2t$

$\omega = \omega_0$ Resonance

$$x'' + \omega_0^2 x = 2\cos 2t$$

$$\boxed{x'' + 4x = 2\cos 2t, \quad x(0) = 0 = x'(0)}$$

General Soln of homog ODE: $x_h(t) = C_1 \cos 2t + C_2 \sin 2t$

$x_1(t) = \cos 2t \quad x_2(t) = \sin 2t$

$W(x_1, x_2) = 2.$

Particular Soln:

- $u_1' = -\frac{g x_2}{W} = -\frac{2\cos 2t \sin 2t}{2} = -\cos 2t \sin 2t$

$u_1(t) = \frac{1}{4}\cos^2 2t$

- $u_2' = \frac{g x_1}{W} = \cos^2 2t = \frac{1}{2}(1 + \cos 4t)$

$u_2(t) = \frac{1}{2}t + \frac{1}{8}\sin 4t$

$x_p(t) = u_1 x_1 + u_2 x_2 = \frac{1}{4}\cos^2 2t \cos 2t + \frac{1}{2}t \sin 2t + \frac{1}{8}\sin 2t \sin 4t.$

Use $\sin 4t = 2\sin 2t \cos 2t$

$x_p(t) = \frac{1}{2}t\sin 2t + \frac{1}{4}(\underbrace{\cos^2 2t + \sin^2 2t}_{=1})\cos 2t$

$= \frac{1}{2}t\sin 2t + \frac{1}{4}\cos 2t$ ← solves homog ODE

General: $x(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{2}t\sin 2t$

Impose Initial Cond: $C_1 = 0$ & $C_2 = 0$ (compute $x'(t)$ at $t=0$)

$x'(t) = 2C_2\cos 2t + \frac{1}{2}\sin 2t + t\cos 2t$

$x'(0) = 2C_2 = 0$

$$\boxed{x(t) = \frac{1}{2}t\sin 2t.}$$