Hints for Solutions of Practice Test #1

1. $mv' = -mg - \gamma v$.

a) $v' + \frac{\gamma}{m} v = -g$ either integrating factor $\mu = e^{\gamma/m\, t}$ or separate

$$\frac{dv}{g + \frac{\gamma}{m} v} = -dt \quad \text{or} \quad (\mu v)' = \mu g$$

$$\left.\begin{aligned} u &= g + \tfrac{\gamma}{m} v \\ du &= +\tfrac{\gamma}{m}\, dv \end{aligned}\right\} \quad +\frac{m}{\gamma} \log\left(g + \tfrac{\gamma}{m} v\right) = -t + C$$

$$\log\left(g + \tfrac{\gamma}{m} v\right) = -\frac{\gamma t}{m} + C$$

$$\boxed{v(t) = \frac{mg}{\gamma} + Ce^{-\frac{\gamma}{m} t}}$$

b) $$v(t) = \frac{10\,kg \cdot 10\, m/s^2}{2\, kg/s} + Ce^{-t/5} \qquad \frac{\gamma}{m} = \frac{1}{5} \frac{1}{sec}$$

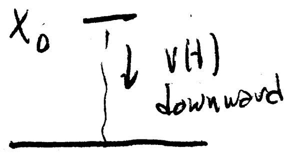
$$v(t) = 50 \frac{m}{s} + Ce^{-t/5} \qquad v(0) = 0$$

$$C = -50$$

$$v(t) = 50(1 - e^{-t/5}) \qquad 50\, m/s \text{ is the terminal vel.}$$

c) $$v(t) = \frac{dx}{dt} = -\left[50 - 50e^{-t/5}\right]$$ Negative because it is directed downward

300m = x_0



$$x(t) = -50t - 250e^{-t/5} + x_0, \qquad x(t=0) = 300$$

$$300 = -250 + x_0 \Rightarrow x_0 = 550$$

$$x(t) = -50t - 250e^{-t/5} + 550$$

$$x(t=0) = 300 \qquad x(T) = 0 \Rightarrow 50(T + 5e^{-T/5}) = 550$$

$$T + 5e^{-T/5} = 11$$

2. $u = y'$ so $u' + 6tu = 6t$ $\quad \mu = e^{\int p(t)dt} = e^{3t^2}$

$$(u\mu)' = \int 6te^{3t^2} dt \qquad w = 3t^2 \quad dw = 6t\, dt$$

p1-2

$e^{3t^2} u = e^{3t^2} + C$

$u = 1 + Ce^{-3t^2}$ $\quad u(0) = 1$ so $C = 0$.

$u(t) = 1 = y'(t)$

$y(t) = t + C$ $\quad y(0) = 1 \Rightarrow C = 1$.

$y(t) = t + 1$ $\quad y'(t) = 1$ $\quad y''(t) = 0$

so $y'' + 6ty' = 0 + 6t = 6t$ ✓

$y(0) = 1$ $\quad y'(t) = 1$.

3. $\frac{dS}{dt} = rS(t)$ $\quad S(t=0) = r$ $\quad S(t) = S_0 e^{rt}$

simple exponential growth.

T: $\quad 2S_0 = S(T) = S_0 e^{rT}$

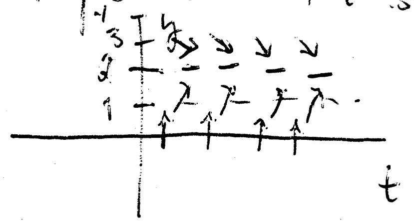
$\ln 2 = rT$ so $T = \frac{1}{r}\ln 2$.

The bigger r is the smaller T is

If $T = 8$, $r = \frac{1}{8}\ln 2$.

4 $y' = f(t,y)$ $\quad f(t,y) = 4 - 2y$ is the direction field.

It is independent of t so the graphical image is



$y = 2$ is a zero $f(t,2) = 0$

$y < 2$ flow into $y = 2$, pos slope

$y > 2$ flow into $y = 2$, neg slope

$\lim_{t\to\infty} y(t) = 2$.

5. 5g/l $\rightarrow$ 1 l/m $\quad$ [100 l] $\rightarrow$ 1 l/m

$\mu = e^{t/100}$

$\frac{dQ}{dt} = 5\frac{g}{l} \cdot 1\frac{l}{m} - \frac{Q}{100}\frac{g}{l} \cdot 1\frac{l}{m}$

$\frac{dQ}{dt} = 5 - \frac{1}{100}Q$.

$(\mu Q) = \int 5e^{t/100} = 500e^{t/100} + C$

$Q(t) = 500 + Ce^{-t/100}$ $\quad Q(0) = 0$ so $C = -500$

$Q(t) = 500(1 - e^{-t/100})$ $\quad Q(100\ln 2) = 250$ l.

6. Put in $y'+py=q$ form: $y' + \left(\frac{3}{t}\right)y = 6t$

$p(t) = \frac{3}{t}$ $\quad q(t) = 6t$ $\quad$ use integrating factor

$\mu(t) = e^{\int p(t)dt} = e^{3\ln t} = e^{\ln(t^3)} = t^3$

so $(\mu y)' = (t^3 y)' = 6t^4$ or $t^3 y = \frac{6}{5}t^5 + c$

$y(t) = \frac{6}{5}t^2 + \frac{c}{t^3}$.

Solve for C if $y(1)=0$. $y(1) = \frac{6}{5} + C = 0$ $\quad C = -6/5$

$$\boxed{y(t) = \frac{6}{5}\left(t^2 - \frac{1}{t^3}\right)}$$

7. Separate Variables: $(3+2y)dy = (3x^2-1)dx$

$3y + y^2 = x^3 - x + C$

We can't solve for y without knowing whether to take the + or - root.

8. $P' = r\left(1-\frac{P}{k}\right)P$. $0 < P(0) < k$. Use partial fractions

$\frac{1}{(1-\frac{P}{k})P} = \frac{A}{1-P/k} + \frac{B}{P} \Rightarrow 1 = AP + B\left(1-\frac{P}{k}\right) = \left(A - \frac{B}{k}\right)P + B$

Coefficients of P & P^0 must match: $A = \frac{B}{k}$ & $B = 1$

$A = 1/k$

Integrate: $\int \frac{dP}{(1-P/k)P} = \frac{1}{k}\int \frac{dP}{1-P/k} + \int \frac{dP}{P} = r\int dt$

so $u = 1 - P/k$ $\quad -1\ln\left|1-\frac{P}{k}\right| + \ln P = rt + C = \ln\left[P(1-P/k)^{-1}\right]$

$du = -\frac{1}{k}dP$

solve: $Ce^{rt} = P(1-P/k)^{-1}$ or $P(t) = Ce^{rt}(1 + C/k\, e^{rt})^{-1}$

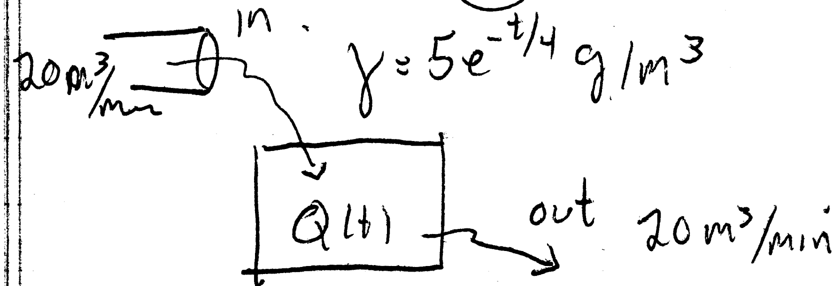
$P(0) = P_0 = C/(1+C/k)$ or $C = P_0(1-P_0/k)^{-1}$ Note $P(0) = P_0 < k$

$$\boxed{P(t) = \frac{P_0 e^{rt}}{1 + P_0/k\,(e^{rt}-1)}}$$

$\lim_{t\to\infty} P(t) = k$ so $0 \le P(t) \le k$ for all t.

(p1-4)

9.



Vol. room = 100 m³
Initial Concentration: 5 g/m³ $\Big\}$ $Q(t=0) = 500\,g$

$$\begin{cases} \dfrac{dQ}{dt} = 20\dfrac{m^3}{min} \times 5e^{-t/4}\dfrac{g}{m^3} - \dfrac{Q(t)}{100}\dfrac{g}{m^3} \times 20\dfrac{m^3}{min} \\ \quad = 100e^{-t/4}\dfrac{g}{m^3} - \dfrac{Q(t)}{5}\dfrac{g}{m^3} \\ Q(t=0) = 500 \end{cases}$$

Solve: $Q' + \frac{1}{5}Q = 100e^{-t/4}$ Integrating factor (can't separate)

$\mu(t) = e^{t/5}$

$(e^{t/5}Q)' = 100e^{t/5}e^{-t/4} = 100e^{-t/20}$

Integrate: $e^{t/5}Q = 100\int e^{-t/20}dt = -2000e^{-t/20} + C$

$Q(t) = -2000e^{3t/20} + e^{-t/5}C$

$Q(0) = -2000 + C = 500 \Rightarrow C = 2500$

$$\boxed{Q(t) = -2000e^{+3t/20} + 2500e^{-t/5}}$$

10. $y' = y$ with $y(0) = 1 \Rightarrow y(t) = e^t$

$y_{n+1} = y_n + hf(t_n, y_n) = y_n + hy_n$

$\boxed{y_{n+1} = (h+1)y_n}$ with $h=1$ $\boxed{y_{n+1} = 2y_n}$

$t_{n+1} = h + t_n = 1 + t_n$ for $h=1$..

	t_n	y_n	e^{t_n}
$n=0$	0	1	1
$n=1$	1	2	$e \approx 2.1\ldots$
$n=2$	2	4	$e^2 > 4$
$n=3$	3	8	$e^3 > 8$
$n=4$	4	16	$e^4 > 16$

etc.