MA214-003 Fall 2009
Test #1 - 100 Points
2 October 2009

INSTRUCTIONS: PLEASE WORK ALL FIVE PROBLEMS BELOW. PLEASE WRITE YOUR NAME ON EACH PAGE. NO CALCULATORS, BOOKS, PAPERS, OR NOTES ARE ALLOWED.

NAME: SOLUTIONS

1. (20 points). Find the solution to the ODE with the given initial condition:

$$y^2(1+x^2)y'(x) = 4x, \quad y(0) = 0.$$

Separate variables:

$$y^2\,dy = \left(\frac{4x}{1+x^2}\right)dx$$

Integrate:

$$\frac{1}{3}y^3 + C = \int \frac{4x}{1+x^2}\,dx = 2\int \frac{du}{u} = 2\ln(1+x^2)$$

$$u = 1+x^2$$
$$du = 2x\,dx$$

$$y^3 = 6\ln(1+x^2) + C$$

Initial cond:

$$y(0) = \sqrt[3]{\ln 1 + C} = 0 \Rightarrow C = 0$$

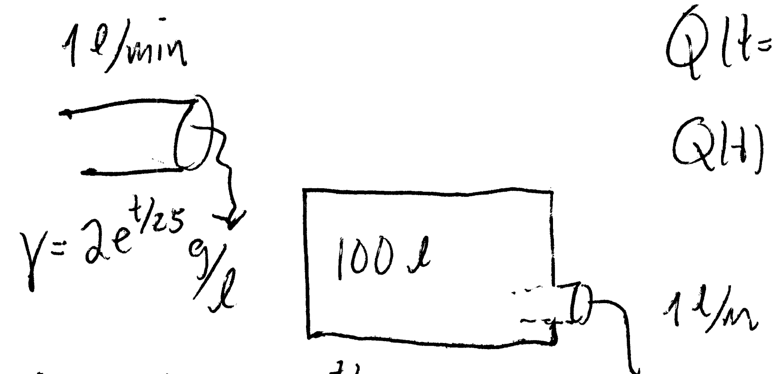
$$y(x) = \left[6\ln(1+x^2)\right]^{\frac{1}{3}}$$

Check:

$$y' = \frac{1}{3}\left[6\ln(1+x^2)\right]^{-2/3} \cdot 6 \cdot \frac{1}{1+x^2} \cdot 2x$$

$$= \frac{4x}{1+x^2} \cdot \frac{1}{y^2} \quad \text{so} \quad y^2y' = \frac{4x}{1+x^2}. \checkmark$$

2. (20 points). A tank initially contains 100 liters of pure water. Polluted water with a concentration of $\gamma = 2e^{t/25}$ grams per liter of mercury is poured into the tank at a rate of one liter per minute. It flows out at the same rate. How much mercury is in the tank after $100 \ln 2$ minutes?



$Q(t=0) = 0$ pure water

$Q(t) \equiv$ amount of mercury in grams

$$\left(\frac{g}{l}\right) \quad \frac{dQ}{dt} = \frac{1\,l}{m} \cdot 2e^{t/25}\,\frac{g}{l} - \frac{Q(t)}{100}\,\frac{g}{l} \times 1\,\frac{l}{min}$$

$$= 2e^{t/25} - \frac{Q}{100} \quad \text{in} \quad \frac{g}{min}$$

$$\frac{dQ}{dt} + \frac{1}{100}Q = 2e^{t/25}$$ integrating factor $\mu(t) = e^{t/100}$

$$(\mu Q)' = \mu q = 2e^{t/100 + t/25} = 2e^{t/20}$$

$$\mu Q = 40e^{t/20} + C \Rightarrow Q(t) = 40e^{t/20 - t/100} + Ce^{-t/100}$$

$$\boxed{Q(t) = 40e^{t/25} + Ce^{-t/100}}$$

Initial Cond: $Q(t=0) = 0 = 40 + C \qquad C = -40$

$$\boxed{Q(t) = 40\left(e^{t/25} - e^{-t/100}\right)}$$

Hg in tank after $100 \ln 2$ min:

$$Q(100\ln 2) = 40\left(e^{4\ln 2} - e^{-\ln 2}\right) = 40\left(16 - \tfrac{1}{2}\right)$$

$$= 20 \cdot 31 = \underline{620\ g.}$$

3. (20 points). Find the unique solution to the initial value problem:

$$y' + 3t^2 y = 4t^2, \quad y(0) = 1.$$

Integrating factor

$$\mu(t) = e^{\int p(t)\,dt} = e^{3\int t^2 dt} = e^{t^3}$$

$$(\mu y)' = \mu q = 4t^2 \cdot e^{t^3}$$

$$\mu y = 4\int t^2 e^{t^3} dt = \frac{4}{3}\int e^u du = \frac{4}{3} e^{t^3} + c$$

$u = t^3$

$du = 3t^2 dt$

$$\boxed{y(t) = \frac{4}{3} + Ce^{-t^3}}$$

Check: $y' = -3t^2 e^{-t^3} \cdot C = -3t^2\left(y - \frac{4}{3}\right) = -3t^2 y + 4t^2$

or $y' + 3t^2 y = 4t^2$.

Initial Cond: $y(0) = \frac{4}{3} + C = 1 \quad C = -\frac{1}{3}$

$$\boxed{y(t) = \frac{1}{3}(4 - e^{-t^3})}$$

4. (20 points). The growth rate of a population $P(t)$ of cave bats changes with time, sometimes positive and sometimes negative. As a model consider the ODE:

$$P'(t) = (\sin t)P(t).$$

a. If the initial population is $P(t=0) = P_0$, find the population at time t.

b. What is the largest size of the population and at what times is it attained?

c. What is the smallest size and when is it attained?

d. Can the population by doubled?

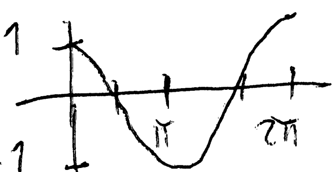
a. $\frac{dP}{dt} = (\sin t)P$ separate: $\frac{dP}{P} = \sin t \, dt$

$\ln P = -\cos t + C$

$\boxed{P(t) = Ce^{-\cos t}}$

Initial Cond: $P(t=0) = Ce^{-1} = P_0$ $\quad C = P_0 e$

$\boxed{P(t) = P_0 e^{1-\cos t}}$



b. largest when $\cos t = -1$ so $t = \pm\pi, \pm 3\pi, \pm 5\pi, \pm 7\pi, \ldots$

odd multiples of π

$\boxed{\begin{array}{l} t = (2n+1)\pi, \; n \in \mathbb{Z} \\ P_{max} = P_0 e^2 \end{array}}$

c. smallest: $\cos t = +1$ so $t = 0, \pm 2\pi, \pm 4\pi, \ldots$ even multiples of π

$\boxed{\begin{array}{l} t = 2n\pi \quad n \in \mathbb{Z} \\ P_{min} = P_0 e \end{array}}$

d. Can $P(t) = 2P_0 = P_0 e^{1-\cos t}$ for some t? YES

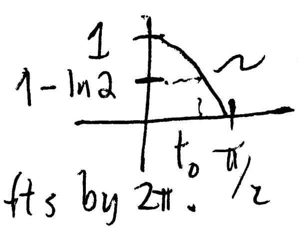
$\boxed{\ln 2 = 1 - \cos t}$ since $0 < \ln 2 < 1$ & $\cos t = 1 - \ln 2$

there is a $t_0 \in (0, \pi)$ so

$0 < 1 - \ln 2 < 1$

$\cos t_0 = 1 - \ln 2$

and also t_1 on $(\frac{3\pi}{2}, 2\pi)$ plus shifts by 2π.



5. (20 points). Consider the first-order ODE:

$$y' = 2y,\ y(0) = 1.$$

a. What is the direction field? Sketch the direction field and describe any special characteristics.

b. Find unique solution of this initial value problem.

c. For the ODE $y'(t) = f(t, y)$, use the Euler method to write y_{n+1} in terms of the uniform step size h, and the initial condition (t_0, y_0).

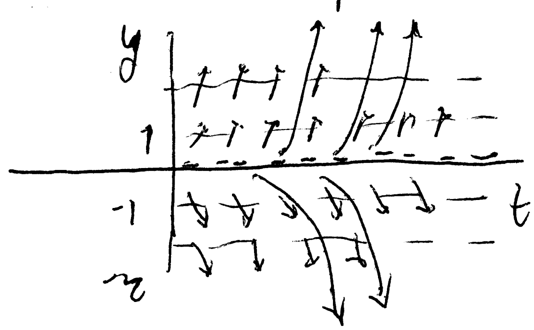
d. Apply the Euler method to the ODE above $y' = 2y$ with step size $h = 1$ and initial condition $(0, 1)$. What is the formula for y_{n+1}? Make a table with three columns labeled by t_j, y_j, and $y(t_j)$ (the exact solution) for $j = 0, 1, 2, 3$. Compare the Euler values y_j with the exact solution $y(t_j)$ (estimate these).

a. $f(t,y) = 2y$ is the direction field. It is independent of t and vanishes at $y=0$.

$y > 0 \quad f > 0$

$y < 0 \quad f < 0$

Solution curves move rapidly away from $y=0$



b. $\frac{dy}{y} = 2 \Rightarrow y(t) = y_0 e^{2t} \qquad y(0) = 1 \Rightarrow y_0 = 1$

$\boxed{y(t) = e^{2t}}$

c. Euler: $y_{n+1} = y_n + h f(t_n, y_n) \quad n = 0, 1, \ldots$

$y_1 = y_0 + h f(t_0, y_0)$ starts the iteration.

d. $y' = 2y \quad (t_0, y_0) = (0, 1) \quad h = 1 \quad f(t_n, y_n) = 2y_n$

$y_{n+1} = y_n + 1 \cdot 2y_n = 3y_n$

$y_1 = y_0 + 1 \cdot 2y_0 = 3y_0$

$y_2 = y_1 + 1 \cdot 2y_1 = 3y_1 = 3^2 y_0$

$y_3 = y_2 + 1 \cdot 2y_2 = 3^3 y_0$

$y_4 = y_3 + 1 \cdot 2y_3 = 3y_3 = 3^4 y_0$

$\boxed{y_{n+1} = 3^{n+1} y_0} = \boxed{3^{n+1}}$ when $y_0 = 1$.

n	t_j	y_j	$y(t_j)$
0	0	1	1
1	1	3	$e^2 > 4$
2	2	9	$e^4 > 16$
3	3	27	$e^6 > 64$
4	4	81	e^8

The approx. is not good for large t because h is so small and $y(t)$ increases rapidly.