1. (20 points). Find the solution to the ODE with the given initial condition:

\[ y^2(1 + x^2)y'(x) = 4x, \quad y(0) = 0. \]

**Separate variables:**

\[ y^2 \, dy = \left( \frac{y}{1+x^2} \right) \, dx \]

**Integrate:**

\[ \frac{1}{3} y^3 + C = \int \frac{y}{1+x^2} \, dx = 2 \int \frac{du}{u} = 2 \ln(1+x^2) \]

\[ u = 1 + x^2, \quad du = 2x \, dx \]

\[ y^3 = 6 \ln(1+x^2) + C \]

**Initial cond:**

\[ y(0) = \sqrt[3]{\ln 1 + C} = 0 \Rightarrow C = 0 \]

\[ y(x) = \left[ 6 \ln(1+x^2) \right]^{\frac{1}{3}} \]

**Check:**

\[ y' = \frac{1}{3} \left[ 6 \ln(1+x^2) \right]^{-\frac{2}{3}} \cdot 6 \cdot \frac{1}{1+x^2} \cdot 2x \]

\[ = \frac{4x}{1+x^2} \cdot \frac{1}{y^2} \quad \text{so} \quad y^2 y' = \frac{4x}{1+x^2}. \quad \checkmark \]
2. (20 points) A tank initially contains 100 liters of pure water. Polluted water with a concentration of \( \gamma = 2e^{t/25} \) grams per liter of mercury is poured into the tank at a rate of one liter per minute. It flows out at the same rate. How much mercury is in the tank after 100 ln 2 minutes?

\[
\begin{align*}
Q(t=0) & = 0 \quad \text{pure water} \\
Q(t) & = \text{amount of mercury in grams}
\end{align*}
\]

\[
\begin{align*}
10/\text{min} & \\
\gamma & = 2e^{t/25} \quad \frac{\text{g}}{\text{l}}
\end{align*}
\]

\[
\left( \frac{\partial}{\partial t} \right) Q = \frac{1}{100} \cdot 2e^{t/25} - \frac{Q(t)}{100} \cdot \frac{1}{100} \times \frac{1}{\text{min}}
\]

\[
= 2e^{t/25} - \frac{Q}{100} \quad \text{in} \quad \frac{\text{g}}{\text{min}}
\]

\[
\frac{dQ}{dt} + \frac{1}{100} Q = 2e^{t/25} \quad \text{Integrating factor} \mu(t) = e^{t/100}
\]

\[
(\mu Q)' = \mu Q = 2e^{t/100} + \frac{1}{25} = 2e^{t/25}
\]

\[
\mu Q = 40e^{t/25} + C \implies Q(t) = \frac{40e^{t/25}}{2} \cdot e^{-t/100} + Ce^{-t/100}
\]

Initial cond: \( Q(t=0) = 0 = 40 + C \quad C = -40 \)

\[
Q(t) = 40 \left( e^{t/25} - e^{-t/100} \right)
\]

Hg in tank after \( 100 \ln 2 \) min:

\[
Q(100\ln 2) = 40 \left( e^{4\ln 2} - e^{-1\ln 2} \right) = 40 \left( 16 - \frac{1}{2} \right) = 20 \cdot 31^2 = 620 \text{ g}.
\]
3. (20 points). Find the unique solution to the initial value problem:

\[ y' + 3t^2 y = 4t^2, \quad y(0) = 1. \]

Integrating factor

\[ M(t) = e^{\int 3t^2 dt} = e^{\frac{3}{3} t^3} = e^{t^3} \]

\[ (Mq)' = Mq = 4t^2 \cdot e^{t^3} \]

\[ Mq = 4 \int t^2 e^{t^3} dt = \frac{4}{3} \int e^u du = \frac{4}{3} e^{t^3} + C \]

\[ u = t^3 \]
\[ du = 3t^2 dt \]

\[ \boxed{y(t) = \frac{4}{3} + Ce^{t^{-3}}} \]

Check:
\[ y' = -3t^2 e^{-t^3}, \quad C = -3t^2 \left( \frac{4}{3} + C \right) = -3t^2 y + 4t^2 \]
\[ \text{or} \quad y' + 3t^2 y = 4t^2. \]

Initial cond: \[ y(0) = \frac{4}{3} + C = 1 \quad C = -\frac{1}{3} \]

\[ \boxed{y(t) = \frac{4}{3} \left( 1 - e^{-t^3} \right)} \]
4. (20 points). The growth rate of a population \( P(t) \) of cave bats changes
with time, sometimes positive and sometimes negative. As a model consider
the ODE:

\[ P'(t) = (\sin t)P(t). \]

a. If the initial population is \( P(t = 0) = P_0 \), find the population at time \( t \).

b. What is the largest size of the population and at what times is it attained?

c. What is the smallest size and when is it attained?

d. Can the population by doubled?
5. (20 points). Consider the first-order ODE:

\[ y' = 2y, \ y(0) = 1. \]

a. What is the direction field? Sketch the direction field and describe any special characteristics.

b. Find unique solution of this initial value problem.

c. For the ODE \( y'(t) = f(t, y) \), use the Euler method to write \( y_{n+1} \) in terms of the uniform step size \( h \), and the initial condition \( (t_0, y_0) \).

d. Apply the Euler method to the ODE above \( y' = 2y \) with step size \( h = 1 \) and initial condition \( (0,1) \). What is the formula for \( y_{n+1} \)? Make a table with three columns labeled by \( t_j \), \( y_j \), and \( y(t_j) \) (the exact solution) for \( j = 0, 1, 2, 3 \). Compare the Euler values \( y_j \) with the exact solution \( y(t_j) \) (estimate these).

\[ f(t, y) = 2y \] is the direction field. It is independent of \( t \) and vanishes at \( y = 0 \).

\[ y > 0 \quad f > 0 \]

\[ y < 0 \quad f < 0 \]

Solution curves move rapidly away from \( y = 0 \)

\[ \frac{dy}{dt} = 2y \] \( \implies y(t) = y_0 e^{2t} \)

\[ y(0) = 1 \implies y_0 = 1 \]

\[ y_1 = y_0 + h f(t_0, y_0) \] starts the iteration.

\[ y' = 2y \] \( (t_0, y_0) = (0, 1) \) \( h = 1 \)

\[ f(t_n, y_n) = 2y_n \]

\[ y_{n+1} = y_n + h f(t_n, y_n) \]

\[ y_n+1 = y_n + 1 \cdot 2y_n = 3y_n - 2y_0 \]

\[ y_2 = y_1 + 1 \cdot 2y_0 = 3y_1 = 3^2 \]

\[ y_3 = y_2 + 1 \cdot 2y_0 = 3^2 \]

\[ y_4 = y_3 + 1 \cdot 2y_0 = 3^2 \]

\[ y_n+1 = 3^{n+1} y_0 = 3^{n+1} \] when \( y_0 = 1 \).