MA214 F09

Some Hints / Solutions on the Practice Problems for test #2

1\. a) $W(y_1, y_2)(x) = \begin{vmatrix} x & xe^x \\ 1 & e^x(1+x) \end{vmatrix} = x^2e^x$

at $x = 1$ this is not zero so independent.

b) Check that y_1 & y_2 solve the ODE by substitution. $y_1(x) = x \quad y_1'(x) = 1 \quad y_1''(x) = 0$

so $x^2 y_1'' - x(x+2)y' + (x+2)y = -(x+2)x - (x+2)x = 0$

2\. Compute $W(w_1, w_2)(t) = \begin{vmatrix} t^2+t & t^2 \\ 2t+1 & 2t \end{vmatrix} = t^2$ & thus is not zero at $t = 1$.

3\. $y'' - ty' = t$. $y = y_p + y_h$. Let $u = y'$ so $u' - tu = t$

Integrating factor $\mu = \exp\{\int -t\,dt\} = e^{-t^2/2}$

$(\mu u)' = \mu g$ so $\mu u = \int te^{-t^2/2} = -\int e^w dw = -e^{-t^2/2} + C$

$w = -t^2/2 \quad dw = -t\,dt$

$u(t) = y'(t) = -1 + e^{t^2/2}C$. Integrate again.

Leave $\int e^{t^2/2}dt$ as it is!

4\. $y'' + 5y' - 6y = 0 \Rightarrow r^2 + 5r - 6 = (r+6)(r-1) = 0$

$y_1(t) = e^{-6t}, \; y_2(t) = e^t \qquad y(t) = C_1e^{-6t} + C_2e^t$

$y'(t) = -6C_1e^{-6t} + C_2e^t$

Initial Cond: $y(0) = C_1 + C_2 = 1$ & $y'(0) = -6C_1 + C_2 = 0$

$C_1 = \frac{1}{7} \quad C_2 = 6/7$.

5\. $(\mathcal{L}f)(s) = \lim_{M\to\infty} \int_0^M e^{-st}(e^{2t}t^2)dt = \lim_{M\to\infty}\int_0^M e^{-(s-2)t}t^2dt$ $(s > 2)$

This can be done by Integration by parts twice or note it is the LT of t^2, $\frac{2}{s^3}$, evaluated at $s-2$: $(\mathcal{L}f)(s) = \frac{2}{(s-2)^3}$

6. Put in standard form: $y'' + \frac{3}{2t} y' - \frac{1}{2t^2} y = 0$, $t > 0$

$W(t) = Ce^{-\int p(t)dt}$ $\quad p(t) = \frac{3}{2t}$ $\quad \int p(t) = \frac{3}{2}\ln t = \ln t^{3/2}$

$= Ce^{-\ln t^{3/2}} = \frac{C}{t^{3/2}}$. We don't know C.

7. Find y_1 & y_2

solving: $y'' + 4y' + 4y = 0$: $r^2 + 4r + 4 = (r+2)^2 = 0$

$y_1(t) = e^{-2t}$, $y_2(t) = te^{-2t}$ Repeated real root!

$$W(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t}(1-2t) \end{vmatrix} = e^{-4t} \neq 0.$$

y_p: $\quad u_1' = -\frac{y_2 g}{W} = \frac{-te^{-2t} \cdot t^{-2} e^{-2t}}{e^{-4t}} = -\frac{1}{t} \Rightarrow u_1(t) = -\ln t$

$u_2' = \frac{y_1 g}{W} = \frac{e^{-2t} \cdot t^{-2} e^{-2t}}{e^{-4t}} = \frac{1}{t^2} \Rightarrow u_2(t) = -\frac{1}{t}$

$$y_p(t) = (-\ln t)e^{-2t} + \left(-\frac{1}{t}\right)te^{-2t} = -e^{-2t}\ln t \underbrace{- e^{-2t}}_{\text{soln to homog ODE}},$$

$$\boxed{y(t) = C_1 e^{-2t} + C_2 te^{-2t} - e^{-2t}\ln t}$$

8. Use the formulas: $2\left(s^2(\mathcal{L}y)(s) - sy(0) - y'(0)\right) - 3\left(s(\mathcal{L}y)(s) - y(0)\right)$
$+ (\mathcal{L}y)(s) = 2 \cdot \frac{2}{4+s^2} = \frac{4}{4+s^2}$ or $(2s^2 - 3s + 1)(\mathcal{L}y)(s) - 2s - 2 + 3$
$= 4(4+s^2)^{-1}$

$$\boxed{(\mathcal{L}y)(s) = \frac{2s-1}{2s^2-3s+1} + \frac{4}{(4+s^2)(2s^2-3s+1)}}.$$

9. $r^2 + 2r + 2r = 0$ $\quad b^2 - 4ac = 4 - 4 \cdot 1 \cdot 2 = -4$, roots: $\frac{-2 \pm 2i}{2}$
$= -1 \pm i$, $e^{(-1 \pm i)t} = e^{-t}(\cos t \pm i \sin t)$

$y_1(t) = e^{-t}\cos t$, $y_2(t) = e^{-t}\sin t$ 2 indep real solns.

Guess: $y_p(t) = At + B$ so $y_p'(t) = A$, $y_p''(t) = 0$

Substitute: $y_p'' + 2y_p' + 2y_p = 2A + 2(At + B) = 4t$

$A = 2$ & $B = -2$ $\quad y_p(t) = 2(t-1)$.