Test #2 - 100 points

20 November 2009

NAME: Solutions

INSTRUCTIONS: PLEASE WORK ALL FIVE PROBLEMS BELOW. PLEASE WRITE YOUR NAME ON EACH PAGE. NO CALCULATORS, BOOKS, PAPERS, OR NOTES ARE ALLOWED.
Each problem is worth 20 points.

1. Find the unique solution to the initial value problem:

$$2y''(t) + 2y'(t) + y(t) = 0, \quad y(0) = 1, y'(0) = 0.$$

Check that you have a pair of independent solutions.

$2r^2 + 2r + 1 = 0 \qquad \frac{-2 \pm \sqrt{4 - 4(2)(1)}}{2 \cdot 2} = -\frac{1}{2} \pm i\frac{1}{2}$ roots.

2 real solns:

$$y_1(t) = e^{-t/2}\cos(t/2)$$
$$y_2(t) = e^{-t/2}\sin(t/2)$$

Wronskian:

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-t/2}\cos(t/2) & e^{-t/2}\sin(t/2) \\ e^{-t/2}\left(-\frac{1}{2}\sin(t/2) - \frac{1}{2}\cos(t/2)\right) & e^{-t/2}\left(\frac{1}{2}\cos\frac{t}{2} - \frac{1}{2}\sin\frac{t}{2}\right) \end{vmatrix}$$

$$= e^{\cdot}\left\{\frac{1}{2}\cos^2(t/2) - \frac{1}{2}\cos(t/2)\sin(t/2) + \frac{1}{2}\sin^2(t/2) + \frac{1}{2}\sin(t/2)\cos(t/2)\right\}$$

$= \frac{1}{2}e^{-t}$. never zero so y_1 & y_2 are indep.

General Soln: $y(t) = C_1 e^{-t/2}\cos(t/2) + C_2 e^{-t/2}\sin(t/2)$

$$y'(t) = -\frac{C_1}{2}e^{-t/2}\cos(t/2) - \frac{1}{2}C_1 e^{-t/2}\sin(t/2)$$
$$-\frac{1}{2}C_2 e^{-t/2}\sin(t/2) + \frac{1}{2}C_2 e^{-t/2}\cos(t/2)$$

Initial Cond:

$y(0) = C_1 = 1.$

$y'(0) = -\frac{1}{2} + \frac{1}{2}C_2 = 0 \Rightarrow C_2 = 1$

Unique Soln:

$$\boxed{y(t) = e^{-t/2}\left[\cos(t/2) + \sin(t/2)\right]}$$

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2. Find the Laplace transform of the unique solution of the initial value problem:

$$y''(t) + 2y'(t) + 3y(t) = 3t^2, \quad y(0) = 1, \quad y'(0) = 2.$$

$$s^2(\mathcal{L}y)(s) - sy(0) - y'(0) + 2\big(s(\mathcal{L}y)(s) - y(0)\big) + 3(\mathcal{L}y)(s)$$

$$= 3 \cdot \frac{2}{s^3}$$

Regroup: $(s^2 + 2s + 3)(\mathcal{L}y)(s) \underbrace{- s - 2 - 2}_{\text{from the initial conditions}} = \frac{6}{s^3}$

$$(s^2+2s+3)(\mathcal{L}y)(s) = \frac{6}{s^3} + (s+4)$$

$$\boxed{(\mathcal{L}y)(s) = \frac{6}{s^3(s^2+2s+3)} + \frac{s+4}{s^2+2s+3}}.$$

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3. Consider a driven, undamped harmonic oscillator:

$$y''(t) + \omega_0^2 y(t) = F_0 t,$$

where F_0 and ω_0 are constants. Find the unique solution to the initial value problem with conditions:

$$y(0) = 0, \quad y'(0) = 1.$$

Homog. ODE: $y'' + \omega_0^2 y = 0$

$y_1(t) = \cos \omega_0 t \qquad y_2(t) = \sin \omega_0 t$

$y_h(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$

Particular Soln: Guess: $At + B = y_p(t)$

so $y_p'(t) = A \quad y_p''(t) = 0$.

Substitute:

$0 + A\omega_0^2 t + B\omega_0^2 = F_0 t$

$B = 0, \quad A = F_0/\omega_0^2$

$y_p(t) = \frac{F_0}{\omega_0^2} t$

Initial Cond:

$y(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{\omega_0^2} t$

$y'(t) = -C_1 \omega_0 \sin \omega_0 t + C_2 \omega_0 \cos \omega_0 t + F_0/\omega_0^2$

so $y(0) = C_1 = 0$

$y'(0) = C_2 \omega_0 + \frac{F_0}{\omega_0^2} = 1 \qquad C_2 = \frac{\omega_0^2 - F_0}{\omega_0^3}$

so $\boxed{y(t) = \frac{\omega_0^2 - F_0}{\omega_0^3} \sin \omega_0 t + \frac{F_0}{\omega_0^2} t}$

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4. Find the most general solution to the ODE:

$$y''(t) - y'(t) - 6y(t) = te^{3t}.$$

Homog. ODE: $r^2 - r - 6 = (r+2)(r-3) = 0$

$y_1(t) = e^{-2t}$ $y_2(t) = e^{3t}$

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & e^{3t} \\ -2e^{-2t} & 3e^{3t} \end{vmatrix} = 5e^{t}$$

$$y_h(t) = C_1 e^{-2t} + C_2 e^{3t}$$

Particular Soln: $u_1' = -\frac{y_2 g}{W} = \frac{-e^{3t} \cdot te^{3t}}{5e^{t}} = -\frac{1}{5} t e^{5t}$

$$\int te^{5t}\,dt = \frac{1}{5}te^{5t} - \frac{1}{5}\int e^{5t}\,dt = \frac{1}{5}te^{5t} - \frac{1}{25}e^{5t}$$

$u = t$ $dv = e^{5t}dt$

$du = dt$ $v = \frac{1}{5}e^{5t}$

$$u_1(t) = -\frac{1}{25}te^{5t} + \frac{1}{125}e^{5t}$$

$$u_2' = \frac{y_1 g}{W} = \frac{e^{-2t}\, te^{3t}}{5e^{t}} = \frac{1}{5}t \qquad u_2(t) = \frac{1}{10}t^2$$

$$y_p(t) = -\frac{1}{25}te^{3t} + \underbrace{\frac{1}{125}e^{3t}}_{\text{soln to homog ODE}} + \frac{1}{10}t^2e^{3t}$$

$$\boxed{y(t) = C_1e^{-2t} + C_2e^{3t} - \frac{1}{25}te^{3t} + \frac{1}{10}t^2e^{3t}}$$

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5. Find the unique solution to the initial value problem:

$$ty''(t) - y'(t) = 3t^2, \quad y(1) = 0, \quad y'(1) = 0.$$

Hint: Use reduction of order or an intelligent guess.

Let $u(t) = y'(t)$ to get a 1st order ODE

$tu' - u = 3t^2$ or $u' + \left(-\frac{1}{t}\right)u = 3t$

$\int p(t)\,dt = -\ln t$ Integrating factor $\mu = e^{\int p} = \frac{1}{t}$

$(\mu u)' = \left(\frac{1}{t}u\right)' = \mu q = 3$ so $\frac{1}{t}u = 3t + C_1$

$u(t) = y'(t) = 3t^2 + C_1 t$

$\boxed{y(t) = t^3 + C_1 t^2 + C_2}$ general soln.

Initial Condition:

$y'(t) = 3t^2 + 2C_1 t$

$$\begin{cases} y(1) = 1 + C_1 + C_2 = 0 \\ y'(1) = 3 + 2C_1 = 0 \end{cases}$$

$C_1 = -3/2$

$C_2 = 1/2$

$\boxed{y(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}}$