1. (30 points). Consider the following nonhomogeneous, second-order initial value problem:

$$y''(t) + 4y'(t) + 5y(t) = \sin t, \quad y(0) = 0, \; y'(0) = 0.$$

a. Find two independent solutions for the associated homogeneous ODE. (You do not need to compute the Wronskian.)

b. Find a particular solution to the nonhomogeneous ODE.

c. Write the general solution to the ODE.

d. Write the unique solution to the initial value problem.

a. $r^2+4r+5=0$ roots $\frac{-4\pm[16-20]^{\frac{1}{2}}}{2} = -2\pm i$ 2 Complex roots

10 $\boxed{y_1(t) = e^{-2t}\cos t \qquad y_2(t) = e^{-2t}\sin t}$

$W(y_1, y_2) = e^{-4t}$ (if you computed it)

b. Since $g(t) = \sin t$ is indep. of y_1 & y_2, guess

$y_p(t) = A\sin t + B\cos t$

$y_p'(t) = A\cos t - B\sin t$

$y_p''(t) = -A\sin t - B\cos t$

$y_p'' + 4y_p' + 5y_p = \sin t\,(-A-4B+5A) + \cos t\,(-B+4A+5B) = \sin t$

10

so $4A-4B=1$, $4A+4B=0$ $\quad A=\frac{1}{8} \quad B=-\frac{1}{8}$

$\boxed{y_p(t) = \frac{1}{8}(\sin t - \cos t)}$

5 c. $y(t) = C_1 e^{-2t}\cos t + C_2 e^{-2t}\sin t + \frac{1}{8}(\sin t - \cos t)$

5 d. $y'(t) = -2C_1 e^{-2t}\cos t - C_1 e^{-2t}\sin t - 2C_2 e^{-2t}\sin t + C_2 e^{-2t}\cos t + \frac{1}{8}(\cos t + \sin t)$

$y(0) = C_1 - \frac{1}{8} = 0 \qquad C_1 = \frac{1}{8}$

$y'(0) = -2C_1 + C_2 + \frac{1}{8} = -\frac{1}{4} + \frac{1}{8} + C_2 = 0 \qquad C_2 = \frac{1}{8}$

$y(t) = \frac{1}{8}e^{-2t}(\cos t + \sin t) + \frac{1}{8}(\sin t - \cos t)$

2. (30 points). Find the unique solution to the second-order initial value problem using Laplace transforms:

$$y''(t) + 2y'(t) + 5y(t) = \delta(t-3), \quad y(0) = 0 \text{ and } y'(0) = 1.$$

$$(s^2+2s+5)(\mathcal{L}y)(s) - 1 = e^{-3s}$$ 5

$$\boxed{(\mathcal{L}y)(s) = \frac{e^{-3s}}{s^2+2s+5} + \frac{1}{s^2+2s+5}}$$ 5

$$s^2+2s+5 = (s+1)^2 + 2^2$$

① $e^{-3s}H(s)$ w/ $H(s) = \frac{1}{(s+1)^2+2^2} = \frac{1}{2}\frac{2}{(s+1)^2+2^2} \to \frac{1}{2}(\sin 2t)e^{-t}$

10 $\hookrightarrow u_3(t)(\mathcal{L}^{-1}H)(t-3) = \frac{1}{2}u_3(t)(\sin 2(t-3))e^{-(t-3)}$

② $\frac{1}{(s+1)^2+2^2} = \frac{1}{2}\frac{2}{(s+1)^2+2^2} \to \frac{1}{2}(\sin 2t)e^{-t}$

10

$$\boxed{y(t) = \frac{1}{2}e^{-t}\sin 2t + \frac{1}{2}u_3(t)e^{-(t-3)}\sin 2(t-3)}$$

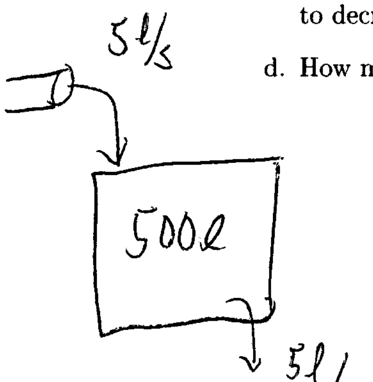
3. (30 points). A 500ℓ tank initially contains 100 grams of pollutant. Polluted water flows into the tank at a rate of $5\ell/s$. The concentration of pollution is $e^{-t/100} g/\ell$. The well-mixed solution flows out of the tank at the same rate.

a. Write an ODE for the quantity of pollutant in the tank at time $t \geq 0$.

b. Final the unique solution of the ODE with the given initial condition.

c. How long does it take until the amount of the pollutant in the tank begins to decrease?

d. How much pollutant is in the tank when $t \to \infty$?



$Q(0) = 100g$ flow in

$$\frac{dQ}{dt} = \left[5\frac{\ell}{s} \cdot e^{-t/100}\frac{g}{\ell}\right] - \frac{Q(t)g}{500\ell} \cdot 5\,\ell/s$$

$$= 5e^{-t/100} - \frac{Q(t)}{100} \quad (g/sec)$$

18

$$\boxed{\frac{dQ}{dt} + \frac{1}{100}Q(t) = 5e^{-t/100}}$$

15

Integrating factor: $\mu(t) = e^{\int p(t)dt} = e^{t/100}$ general soln.

$(Q\mu)' = \mu q = 5$

$\mu Q = 5t + C$

10

$Q(0) = 100 = C$

$$\boxed{Q(t) = 5te^{-t/100} + Ce^{-t/100}}$$

$$\boxed{Q(t) = 5te^{-t/100} + 100e^{-t/100}}$$

2 $\lim_{t\to\infty} Q(t) = 0$. $\left(\lim_{t\to\infty} \frac{t}{e^{t/100}} = \left(\frac{\infty}{\infty}\right) \lim_{t\to\infty} \frac{1}{\frac{1}{100}e^{t/100}} = \lim_{t\to\infty} 100e^{-t/100} = 0\right.$

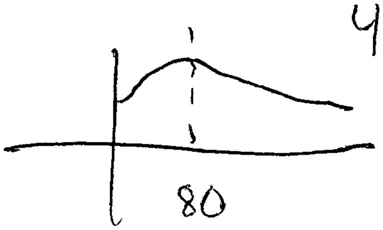
L'Hopital's Rule

at first it increases but then $\to 0$

3 $Q(t) = 5e^{-t/100}(t + 20)$

$Q'(t) = -\frac{1}{20}e^{-t/100}(t+20) + 5e^{-t/100} = e^{-t/100}\left(5 - 1 - \frac{1}{20}t\right) = 0$

$4 = \frac{1}{20}t$ or $\boxed{t = 80}$ sec (critical pt.) 4



4. (30 points). Use the method of Laplace transforms to solve the initial value problem:

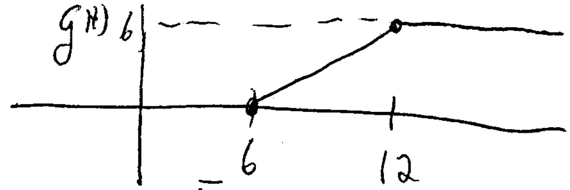
$y''+2y'+2y=g(t)$

$$y''(t)+=g(t), \qquad y(0)=y'(0)=0,$$

with $y(0)=0$ and $y'(0)=0$, where the source term g is given by

$$g(t)=\begin{cases} 0 & 0\le t\le 6 \\ t-6 & 6\le t\le 12 \\ \cancel{2}\,6 & t\ge 12 \end{cases}$$

HINT: Use $As+B$ and $Cs+D$ in the numerators of both terms of the partial fraction expansion.



$g(t)=(t-6)u_6(t)-(t-6)u_{12}(t)$
$\quad+6u_{12}(t)$ 5

$=(t-6)u_6(t)-(t-12)u_{12}(t)$ 5 $\quad (\mathcal{L}g)(s)=\dfrac{e^{-6s}}{s^2}-\dfrac{e^{-12s}}{s^2}$

ODE $\Rightarrow \quad (s^2+2s+2)(\mathcal{L}y)(s)=(\mathcal{L}g)(s)$

$\boxed{(\mathcal{L}y)(s)=e^{-6s}H(s)\overset{5}{-}e^{-12s}H(s)}, \quad H(s)=\dfrac{1}{s^2(s^2+2s+2)}$

Partial frac.: $\dfrac{1}{s^2(s^2+2s+2)}=\dfrac{As+B}{s^2}+\dfrac{Cs+D}{s^2+2s+2}$

5 $\quad 1=(As+B)(s^2+2s+2)+(Cs+D)s^2$
$\quad =(A+C)s^3+(2A+B+D)s^2+(2A+2B)s+2B$

$2B=1$ so $\boxed{B=\tfrac12}$ $\quad C=-A \quad 2A+D=-\tfrac12 \quad A=-B$

$\boxed{C=\tfrac12 \quad D=\tfrac12 \quad A=-\tfrac12}$

$H(s)=\dfrac12\left(\dfrac{-s+1}{s^2}\right)+\dfrac12\left(\dfrac{s+1}{(s+1)^2+1}\right)=-\dfrac12\dfrac1s+\dfrac12\dfrac1{s^2}+\dfrac12\dfrac{s+1}{(s+1)^2+1}$

$(\mathcal{L}^{-1}H)(t)=-\tfrac12+\tfrac12 t+\tfrac12 e^{-t}\cos t$ 5

$\Rightarrow \boxed{\begin{aligned} y(t)=\;&u_6(t)\left(-\tfrac12+\tfrac12(t-6)+\tfrac12 e^{-(t-6)}\cos(t-6)\right) \quad 5 \\ &-u_{12}(t)\left(-\tfrac12+\tfrac12(t-12)+\tfrac12 e^{-(t-12)}\cos(t-12)\right). \end{aligned}}$

5

5. (30 points).

a. Find the unique solution to the initial value problem:

$$\frac{dy}{dt} = 2y(4-y), \quad y(0) = 2.$$

b. Compute $\lim_{t\to\infty} y(t)$.

Separate: $\frac{dy}{y(4-y)} = 2dt$ $\qquad \frac{1}{y(4-y)} = \frac{1}{4}\left(\frac{1}{y} + \frac{1}{4-y}\right)$

5

$\int \frac{1}{4}\left(\frac{1}{y} + \frac{1}{4-y}\right) dy = 2t + C$ $\qquad$ We can assume $0 < y < 4$

$\frac{1}{4}\left(\ln y - \ln|4-y|\right) = 2t + C \Rightarrow \ln\left(\frac{y}{4-y}\right) = 8t + C$

5

5 $\quad \frac{y}{4-y} = Ce^{8t}$

$y = 4Ce^{8t} - Ce^{8t}y$ or $(1+Ce^{8t})y = 4Ce^{8t}$

5 $\quad \boxed{y(t) = \frac{4Ce^{8t}}{1+Ce^{8t}}}$

Initial Value: $y(0) = \frac{4C}{1+C} = 2$ or $4C = 2C + 2$

$C = 1$

5

$\boxed{y(t) = \frac{4e^{8t}}{1+e^{8t}}}$

5

$\lim_{t\to\infty} \left(y(t)\right) = \left(\frac{e^{8t}}{e^{8t}}\right)\left(\frac{4}{1+e^{-8t}}\right) = \underline{\underline{4}}$

6. (30 points). Consider the nonhomogeneous second order ODE.

$$y''(t) + 2y'(t) + y(t) = t^{-2}e^{-t}. \quad t > 0$$

a. Find two independent solutions to the homogeneous ODE. Compute Wronskian of these two solutions.

b. Find a particular solution to the nonhomogeneous ODE using the variation of parameters method.

c. Write the most general solution to the nonhomogeneous ODE.

a $r^2 + 2r + r = 0$ $\quad \frac{-2 \pm (4-4)^{\frac{1}{2}}}{2} = -1$ one double root

$y_1(t) = e^{-t}$ $\quad y_2(t) = te^{-t}$ $\quad W(y_1, y_2)(t) = e^{-2t}(-t+1) + te^{-2t}$

10 $\quad y_1'(t) = -e^{-t}$ $\quad y_2'(t) = e^{-t}(1-t)$ $\quad = +e^{-2t}$

independent because W never vanishes

b. 5 $\quad u_1' = \frac{-y_2 g}{W} = \frac{-te^{-t} \cdot t^{-2}e^{-t}}{e^{-2t}} = -\frac{1}{t}$ $\quad u_1(t) = -\ln t$

5 $\quad u_2' = \frac{y_1 g}{W} = \frac{e^{-t} \cdot t^{-2}e^{-t}}{e^{-2t}} = \frac{1}{t^2}$ $\quad u_2(t) = \frac{-1}{t}$

5 $\quad y_p(t) = (y_1 u_1)(t) + (y_2 u_2)(t) = -e^{-t}\ln t \underbrace{- e^{-t}}_{\text{part of } y_h(t)}$

$y_p(t) = -e^{-t}\ln t$

c. 5 $\quad \boxed{y(t) = -e^{-t}\ln t + C_1 e^{-t} + C_2 te^{-t}}$