MA214-003 Spring 2010
Test #1 100 Points
26 February 2010

INSTRUCTIONS: PLEASE WORK ALL FIVE PROBLEMS BELOW. PLEASE WRITE YOUR NAME ON EACH PAGE. NO CALCULATORS, BOOKS, PAPERS, OR NOTES ARE ALLOWED.

NAME: Solutions

1. (20 points).

(a.) Find two independent solutions of the ODE:

$$y'' - y' - 2y = 0.$$

Verify that the solutions are independent.

(b.) Write the most general solution to the ODE.

(c.) Find the unique solution with initial values: $y(0) = 1$ and $y'(0) = 0$.

a) Characteristic eqn: $r^2 - r - 2 = 0$ $\quad b^2 - 4ac = 1 - 4\cdot1\cdot(-2) = 9$

roots: $r_1 = \frac{1+3}{2} = 2 \qquad r_2 = \frac{1-3}{2} = -1$

2 soln: $y_1(t) = e^{2t} \qquad y_2(t) = e^{-t}$.

$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = e^{t}(-1-2) = -3e^{t} \neq 0$

since W is never zero on $\mathbb{R}$ the 2 solns. are independent

b.) General soln: $y(t) = C_1 y_1(t) + C_2 y_2(t) = C_1 e^{2t} + C_2 e^{-t}$

for part c.): $y'(t) = 2C_1 e^{2t} - C_2 e^{-t}$

c.) $y(0) = 1 = C_1 + C_2 \Rightarrow 1 = 3C_1 \Rightarrow$ $\boxed{C_1 = 1/3, \; C_2 = 2/3}$

$y'(0) = 0 = 2C_1 - C_2 \Rightarrow C_2 = 2C_1$

Unique soln:

$$\boxed{y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}}$$

2. (20 points). The population of squirrels on campus begins with an initial population of 10. The population increases at a rate of 10% per year. The campus hawks eat 5 squirrels per year. Assume the growth and eating are done continuously throughout the year. How long does it take for the squirrel population to go to zero? The ODE satisfied by the population $P(t)$ at time $t \geq 0$ is

$$P'(t) = rP(t) + k.$$

$P(t=0) = 10$ $\quad$ 10% incr $\Rightarrow r = \frac{1}{10}$

$k = -5$ per year (loss so negative)

$$\boxed{\frac{dP}{dt} = \frac{1}{10}P - 5}$$

Separate: $\dfrac{dP}{\frac{1}{10}P - 5} = dt$

Integrate: $10\left(\int \frac{du}{u}\right) = 10 \ln\left(\frac{1}{10}P - 5\right) = t + C$

$u = \frac{1}{10}P - 5$

$du = \frac{1}{10}dP$

$\left(\frac{1}{10}P - 5\right) = Ce^{\frac{1}{10}t}$

$$\boxed{P(t) = 50 + Ce^{\frac{1}{10}t}}$$ General Solution

Impose the initial condition:

$$P(0) = 10 = 50 + C \Rightarrow C = -40$$

$$\boxed{P(t) = 50 - 40e^{\frac{1}{10}t}}$$

When will $P(t) = 0$? Find T so $50 = 40e^{\frac{1}{10}T}$

$5/4 = e^{T/10}$

$$\Rightarrow \boxed{T = 10\ln(5/4)}$$ years.

$5/4 > 1$ so $\ln(5/4) > 0$.

about 4-5 years.

3. (20 points). Find the unique solution to the initial value problem:

$$(1+t^2)y' + 2ty = 3t^2, \quad y(0) = 2.$$

Put into standard form:

$$y' + \frac{2t}{1+t^2}y = \frac{3t^2}{1+t^2} \quad (\text{OK for all } t)$$

Integrating factor:

$$p(t) = \frac{2t}{1+t^2} \qquad \int p(t)\,dt = \int \frac{2t\,dt}{1+t^2} = \ln(1+t^2)$$

$$u = 1+t^2$$
$$du = 2t\,dt$$

$$\mu(t) = e^{\int p(t)\,dt} = e^{\ln(1+t^2)} = (1+t^2)$$

ODE becomes:

$$(\mu y)' = \mu q = (1+t^2)\cdot\left(\frac{3t^2}{1+t^2}\right) = 3t^2$$

Integrate:

$$\mu y = 3\int t^2\,dt = t^3 + C$$

General Solution

$$\boxed{y(t) = \frac{t^3}{1+t^2} + \frac{C}{1+t^2}}$$

You can easily check this:

$$y' = \frac{3t^2}{1+t^2} - \frac{2t^4}{(1+t^2)^2} - \frac{2tC}{(1+t^2)^2}$$

Initial Condition: $y(0) = C = 2$.

$$y(t) = \frac{t^3}{1+t^2} + \frac{2}{1+t^2} = \frac{t^3+2}{1+t^2}$$

$$\boxed{y(t) = \frac{t^3+2}{1+t^2}}$$

4. (20 points). A 500 liter holding pond initially contains 300 liters of pure water. A polluted stream dumps 2 liter per minute of polluted water into the pond. The polluted water has a concentration of 1 gram per liter of pollutant. Water flows out of the pond more slowly at 1 liter per minute.

a. How many minutes does it take until the pond overflows?

b. What is the volume of water in the pond at any time $t \geq 0$?

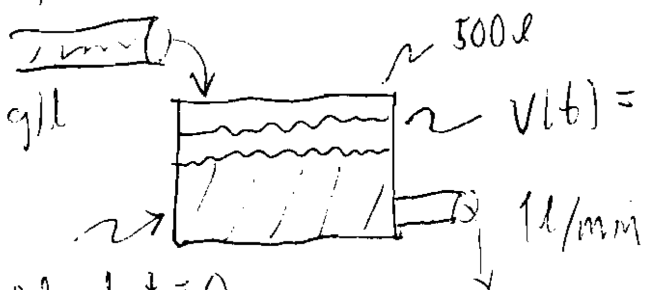
c. Let $Q(t)$ be the amount of pollutant, in grams, in the pond at time $t \geq 0$. Find $Q(t)$. Remember:

$$\frac{dQ}{dt} = [\text{flow in}] - [\text{flow out}].$$

d. How much pollutant is in the pond when it overflows?

2 l/min

1 g/l



$V(t) = 300 + t$

$Q(t=0) = 0$ Initial Condition

300 l at $t = 0$

a) Fluid enters at 2 l/m & leaves at 1 l/m so a net increase of 1 l/min so it takes $\boxed{200 \text{ min}}$ to add 200 l.

b) Volume $\boxed{V(t) = 300 + t}$ since one liter per minute is added. (units: minutes)

concentration in tank at time t.

c.) $\dfrac{dQ}{dt} = 2\dfrac{l}{min} \cdot 1\dfrac{g}{l} - \dfrac{Q(t)}{(300+t)}\dfrac{g}{min} \cdot \dfrac{1 l}{min}$

$$\boxed{\frac{dQ}{dt} = 2 - \frac{Q}{300+t}}$$ in g/min $\quad Q' + \dfrac{Q}{300+t} = 2$

$p(t) = \dfrac{1}{300+t}$ $\quad \int p(t)\,dt = \ln(300+t)$

$\mu = (300+t)$. $\quad \mu Q = \int \mu q = 2\int(300+t)\,dt$

$= 600t + t^2 + C$

$$Q(t) = \frac{600t + t^2 + C}{300+t}$$

$Q(0) = \dfrac{C}{300} = 0 \quad C = 0$

$$\boxed{Q(t) = \frac{600t + t^2}{300+t}}$$

d). $Q(200) = \dfrac{12 \times 10^4 + 4 \times 10^4}{500} = \boxed{320 \text{ gms.}}$

5. (20 points). Consider the first-order ODE:

$$y' = 2t, \ y(0) = 0.$$

a. What is the direction field? Sketch the direction field and describe any special characteristics.

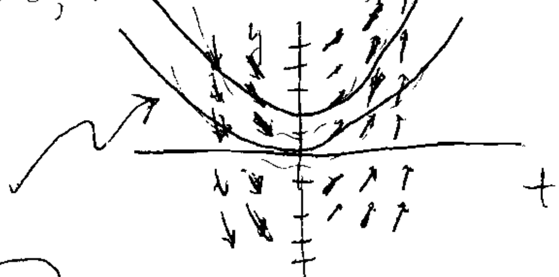
b. Find unique solution of this initial value problem.

c. For the ODE $y'(t) = f(t, y)$, use the Euler method to write y_{n+1} in terms of the uniform step size h, and the initial condition (t_0, y_0).

d. Apply the Euler method to the ODE above $y' = 2t$ with step size $h = 1$ and initial condition $(0, 0)$. What is the formula for y_{n+1}? Make a table with three columns labeled by t_j, y_j, and $y(t_j)$ (the exact solution) for $j = 0, 1, 2, 3$. Compare the Euler values y_j with the exact solution $y(t_j)$.

a.) $\boxed{f(t,y) = 2t}$ independent of y. $f > 0, t > 0$ & $f < 0, t < 0$

$f = 0$ at $t = 0$



b.) $y(t) = t^2 + C$ so the soln curves are parabolic

$y(0) = 0 \Rightarrow C = 0$ $\boxed{y(t) = t^2}$

c.) $y_{n+1} = y_n + h f(t_n, y_n) = y_n + h(2t_n) = y_n + 2ht_n$

$\boxed{y_{n+1} = y_n + 2ht_n}$

d.) $h = 1$ $(t_0, y_0) = (0, 0)$

$\begin{cases} y_1 = y_0 + 2t_0 = 0 \\ t_1 = 1 \end{cases}$

$y_2 = y_1 + 2t_1$

$\begin{cases} y_2 = 2 \\ t_2 = 2 \end{cases}$

$\begin{cases} y_3 = y_2 + 2t_2 \\ \quad = 2 + 2 \cdot 2 = 6 \\ t_3 = 3 \end{cases}$

n	t_n	y_n
0	0	0
1	1	0
2	2	1
3	3	6

Actual soln. $y(3) = 3^2 = 9$. The solution grows so rapidly that the euler approximation isn't good even at $t = 2$!