MA214 Spring 2010
Test #2 - 100 points
9 April 2010

NAME: Solutions

INSTRUCTIONS: PLEASE WORK ALL FOUR PROBLEMS BELOW. PLEASE WRITE YOUR NAME ON EACH PAGE. NO CALCULATORS, BOOKS, PAPERS, OR NOTES ARE ALLOWED.
Each problem is worth 25 points.

1. Consider the initial value problem:

$$y''(t) + 4y'(t) + 5y(t) = 0, \quad y(0) = 0, y'(0) = 1.$$

(a) Find a pair of linearly independent real solutions to the ODE. Be sure to check that they are independent.

(b) Write the most general real solution to the ODE.

(c) Find the unique real solution to the initial value problem.

a.) $r^2 + 4r + 5 = 0$ roots $\frac{-4 \pm [16 - 4 \cdot 5]^{\frac{1}{2}}}{2} = -2 \pm i$

$\boxed{y_1(t) = e^{-2t}\cos t, \quad y_2(t) = e^{-2t}\sin t}$ (obtain from $e^{(-2\pm i)t} = e^{-2t}e^{\pm it}$ and Euler: $e^{-2t}(\cos t \pm i\sin t)$)

$$W(t) = \begin{vmatrix} e^{-2t}\cos t & e^{-2t}\sin t \\ (-2\cos t - \sin t)e^{-2t} & (-2\sin t + \cos t)e^{-2t} \end{vmatrix} = e^{-4t} \neq 0 \text{ so indep.}$$

$\boxed{W(t) = e^{-4t}}$

b.) $y(t) = C_1 e^{-2t}\cos t + C_2 e^{-2t}\sin t$

c.) $y'(t) = -2C_1 e^{-2t}\cos t - C_1 e^{-2t}\sin t - 2C_2 e^{-2t}\sin t + C_2 e^{-2t}\cos t$

$$\left.\begin{aligned} y(0) &= C_1 = 0 \\ y'(0) &= C_2 = 1 \end{aligned}\right\} \boxed{y(t) = e^{-2t}\sin t}$$

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2. Consider the ODE: Find the most general real solution to the ODE:

$$y'' + 2y' + y = t.$$

(a) Find the most general real solution to the ODE.

(b) Find the unique real solution satisfying the initial conditions:

$$y(0) = 0, \;\; y'(0) = 0.$$

a) Homog ODE: $r^2+2r+1 = (r+1)^2 = 0$ $r = -1$ double root.

$$\boxed{\begin{array}{l} y_1(t) = e^{-t}, \quad y_2(t) = te^{-t} \\ y_h(t) = C_1 e^{-t} + tC_2 e^{-t} \end{array}}$$

b.) Guess: $g(t) = t$ indep of y_1 & y_2: $y_p(t) = At+B$

so $y_p'(t) = A$ & $y_p''(t) = 0$. Substitute:

$y_p'' + 2y_p' + y_p = 2A + At + B = t \Rightarrow A = 1, 2A+B = 0$

so $B = -2$ $\boxed{y_p(t) = t-2}$

Check: $y_p'(t) = 1$ $y_p''(t) = 0$: $0 + 2 + t - 2 = t$ ✓

$y(t) = C_1 e^{-t} + C_2 te^{-t} + t - 2$

$y'(t) = -C_1 e^{-t} + C_2 e^{-t} - C_2 te^{-t} + 1$

$y(0) = C_1 - 2 = 0 \Rightarrow C_1 = 2$

$y'(0) = -C_1 + C_2 + 1 = 0$ $C_2 = 1$

$$\boxed{y(t) = 2e^{-t} + te^{-t} + t - 2.}$$

Check $y(0) = 2 - 2 = 0$ ✓

$y'(0) = -2 + 1 + 1 = 0$ ✓

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3. Consider a driven, undamped harmonic oscillator:

$$y''(t) + \omega_0^2 y(t) = F_0 \cos(\omega t),$$

where F_0, ω_0 and ω are constants and $\omega_0 \neq \omega$. Find the unique solution to the initial value problem with conditions:

$$y(0) = 0, \quad y'(0) = 0.$$

Homog ODE: $y'' + \omega_0^2 y = 0 \qquad y_1(t) = \cos\omega_0 t, \quad y_2(t) = \sin\omega_0 t$

$$\boxed{y_h(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t}$$

Particular Soln: $\omega \neq \omega_0$ so guess

$$y_p(t) = A\cos\omega t + B\sin\omega t$$
$$y_p'(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$
$$y_p''(t) = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

Substitute:

$$y_p'' + \omega_0^2 y_p = (\cos\omega t)(-A\omega^2 + A\omega_0^2) + (\sin\omega t)(-B\omega^2 + B\omega_0^2)$$
$$= F_0 \cos\omega t$$

so $B = 0$ & $A(\omega_0^2 - \omega^2) = F_0 \Rightarrow A = \dfrac{F_0}{\omega_0^2 - \omega^2}$

$$\boxed{y_p(t) = \frac{F_0}{\omega_0^2 - \omega^2}\cos\omega t}$$

$$y_p(t) = C_1\cos\omega_0 t + C_2 \sin\omega_0 t + \frac{F_0}{\omega_0^2 - \omega^2}\cos\omega t$$

$$y_p'(t) = -C_1\omega_0 \sin\omega_0 t + C_2\omega_0\cos\omega_0 t - \frac{F_0\omega}{\omega_0^2 - \omega^2}\sin\omega t$$

$$y_p(0) = C_1 + \frac{F_0}{\omega_0^2 - \omega^2} = 0 \qquad C_1 = -\frac{F_0}{\omega_0^2 - \omega^2}$$

$$y_p'(0) = C_2\omega_0 = 0 \Rightarrow C_2 = 0$$

$$\boxed{y(t) = \frac{F_0}{\omega_0^2 - \omega^2}\left[\cos\omega t - \cos\omega_0 t\right]}$$

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4. Find the most general solution to the ODE:

$$y''(t) + y'(t) - 2y(t) = te^t.$$

Homog ODE: $y'' + y' - 2y = 0$

$r^2 + r - 2 = 0 \quad b^2 - 4ac = 1 - (4)(1)(-2) = 9$

roots: $\frac{-1 \pm \sqrt{9}}{2} = -\frac{1}{2} \pm \frac{3}{2} \Rightarrow r_1 = -2 \quad r_2 = 1$

$\boxed{y_1(t) = e^{-2t} \quad y_2(t) = e^t}$

$y_h(t) = C_1 e^{-2t} + C_2 e^t$

Guess $g(t) = te^t$ indep of y_1 & y_2

$y_p(t) = Ate^t$

$y_p'(t) = Ae^t + Ate^t$

$y_p''(t) = 2Ae^t + Ate^t$

$y_p'' + y_p' - 2y_p = (2A - 2A)te^t + (2A + A)e^t \overset{?}{=} te^t$ Does not work.

No soln!

Variation of Parameters:

$W(t) = \begin{vmatrix} e^{-2t} & e^t \\ -2e^{-2t} & e^t \end{vmatrix} = 3e^{-t} \neq 0.$

$u_1'(t) = \frac{-y_2(t)g(t)}{W(t)} = -\frac{e^t \cdot te^t}{3e^{-t}} = -\frac{1}{3}te^{3t} \Rightarrow u(t) = \frac{-1}{3}\int te^{3t}\,dt$

$\int \underbrace{t}_{u}\underbrace{e^{3t}\,dt}_{dv} = \frac{1}{3}te^{3t} - \frac{1}{3}\int e^{3t}\,dt = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t}$

$v = \frac{1}{3}e^{3t}$

so $u_1(t) = -\frac{1}{9}te^{3t} + \frac{1}{27}e^{3t}$

$u_2'(t) = \frac{y_1(t)g(t)}{W(t)} = \frac{e^{-2t}te^t}{3e^{-t}} = \frac{1}{3}t \Rightarrow u_2(t) = \frac{1}{6}t^2$

$y_p(t) = y_1(t)u_1(t) + y_2(t)u_2(t) = -\frac{1}{9}te^t + \underbrace{\frac{1}{27}e^t}_{\text{in } y_h(t)} + \frac{1}{6}t^2e^t$

$\boxed{y_g(t) = C_1e^{-2t} + C_2e^t - \frac{1}{9}te^t + \frac{1}{6}t^2e^t}$