**MA214-002 Spring 2011**
**Final Exam - 180 Points**
**4 May 2011, 8:00–10:00 AM, CB 339**

INSTRUCTIONS: PLEASE WORK ALL 6 PROBLEMS BELOW. PLEASE WRITE YOUR NAME AND SECTION NUMBER ON EACH PAGE. NO CALCULATORS, BOOKS, PAPERS, OR NOTES ARE ALLOWED.

**LAPLACE TRANSFORM FORMULAS ON THE LAST PAGE.**

**NAME:** Solutions

| PROBLEM | MAXIMUM GRADE | SCORE |
|---|---|---|
| 1 | 30 | |
| 2 | 30 | |
| 3 | 30 | |
| 4 | 30 | |
| 5 | 30 | |
| 6 | 30 | |
| TOTAL | 180 | |

1. (30 points). Find the unique solution to the first-order ODE:

$$y\frac{dy}{dx} = (1+x)(1+y^2), \quad y(0) = 1, \quad y \geq 0.$$

Separate Variables:

$$\frac{y}{1+y^2}dy = (1+x)dx \quad 5$$

Integrate: $\int \frac{y}{1+y^2}dy = \overset{5}{\frac{1}{2}\ln(1+y^2) + c} = \overset{5}{x + \frac{1}{2}x^2 + d}$

$u = 1+y^2$

$du = 2y\,dy$

$$\ln(1+y^2) = 2x + x^2 + f$$

$$1+y^2 = Ce^{2x+x^2}$$

$$(*) \quad 5 \quad \boxed{y^2 = Ce^{2x+x^2} - 1}$$

Initial Condition:

$y(0) = 1$ so $y^2(0) = 1 = C - 1 = 1 \Rightarrow C = 2$.

5

Since we are told $y > 0$:

$$5 \quad \boxed{y(x) = \left[2e^{2x+x^2} - 1\right]^{\frac{1}{2}}}$$

Check: $y' = \frac{1}{2}\left[2e^{2x+x^2} - 1\right]^{-\frac{1}{2}}\left(2(2+2x)e^{2x+x^2}\right)$

$$yy' = 2(1+x)e^{2x+x^2} = (1+x)(2e^{2x+x^2})$$

and we see from $(*)$ that

$$y^2 + 1 = 2e^{2x+x^2}$$

so

$$yy' = (1+x)(1+y^2). \checkmark \quad 2$$

2. (30 points).

Consider the following nonhomogeneous, second-order initial value problem:

$$y''(t) + 4y'(t) + 5y(t) = \sin t, \quad y(0) = 0, \ y'(0) = 0.$$

a. Find two independent solutions for the associated homogeneous ODE. (You do not need to compute the Wronskian.)

b. Find a particular solution to the nonhomogeneous ODE.

c. Write the general solution to the ODE.

d. Write the unique solution to the initial value problem.

a) $y'' + 4y' + 5y = 0$ $\quad r^2 + 4r + 5 = 0$ $\quad r_\pm = \dfrac{-4 \pm [16-20]^{\frac{1}{2}}}{2}$

$= -2 \pm i$

5 $\boxed{y_1(t) = e^{-2t}\cos t}$

5 $\boxed{y_2(t) = e^{-2t}\sin t}$

b.) Since $g(t) = \sin t$ is independent of $y_1$ & $y_2$, guess

5 $y_p(t) = A\cos t + B\sin t$

$y_p'(t) = -A\sin t + B\cos t$

$y_p''(t) = -A\cos t - B\sin t$

$y_p'' + 4y_p' + 5y_p = (\cos t)(-A + 4B + 5A) + (\sin t)(-B - 4A + 5B) = \sin t$

$\Rightarrow$ $0 = 4(A+B)$ and $1 = -4A + 4B$ or $B = \frac{1}{8}$ $\quad A = -\frac{1}{8}$

5 $\boxed{y_p(t) = -\frac{1}{8}\cos t + \frac{1}{8}\sin t}$ (easy to check!)

c.) $\boxed{y_{gen}(t) = C_1 e^{-2t}\cos t + C_2 e^{-2t}\sin t - \frac{1}{8}\cos t + \frac{1}{8}\sin t}$

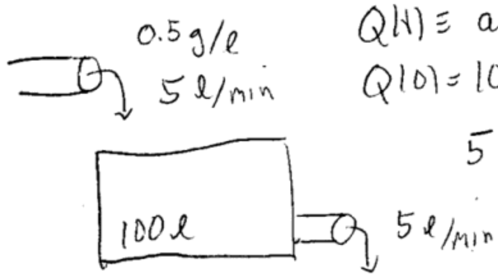
5 $y_g'(t) = -2C_1 e^{-2t}\cos t - C_1 e^{-2t}\sin t - 2C_2 e^{-2t}\sin t + C_2 e^{-2t}\cos t + \frac{1}{8}\sin t + \frac{1}{8}\cos t$ .

d) $y_g(0) = C_1 - \frac{1}{8} = 0 \Rightarrow C_1 = \frac{1}{8}$ 3

5 $y_g'(0) = -2C_1 + C_2 + \frac{1}{8} = 0 \Rightarrow -\frac{1}{4} + \frac{1}{8} + C_2 = -\frac{1}{8} + C_2 = 0 \Rightarrow C_2 = \frac{1}{8}$

$\boxed{y_g(t) = \frac{1}{8}\left[e^{-2t}\cos t + e^{-2t}\sin t - \cos t + \sin t\right]}$

3. (30 points). A 100 liter tank initially contains 100 grams of salt. A solution of water and salt with a concentration of 0.5 grams per liter flows into the tank at a rate of 5 liters per minute. It is thoroughly mixed in the tank and flows out at the same rate. Find the amount of salt in the tank as a function of time. How much salt is in the tank as time goes to infinity?




$Q(t) \equiv$ amount of salt in grms in tank

$Q(0) = 100g$

5 $$\frac{dQ}{dt} = \underbrace{\left(0.5\frac{g}{l}\right)\left(5\frac{l}{min}\right)}_{\text{flow in}} - \underbrace{\frac{Q(t)}{100}\cdot 5}_{\text{flow out}}$$

5 $$= 2.5 - \frac{1}{20}Q \quad \left(\frac{g}{min}\right)$$

$$\frac{dQ}{dt} = 2.5 - \frac{1}{20}Q$$

Separate variables: 5 $$\frac{dQ}{2.5 - \frac{1}{20}Q} = dt \Rightarrow -20\int\frac{du}{u} = \int dt$$

$u = 2.5 - \frac{1}{20}Q$

$du = -\frac{1}{20}dQ$

5 $$-20\ln u = t + C$$

$$\ln\left|2.5 - \frac{1}{20}Q\right| = -\frac{1}{20}t + C$$

$$2.5 - \frac{1}{20}Q = Ce^{-\frac{1}{20}t}$$

5 $$\boxed{50 - Ce^{-\frac{1}{20}t} = Q(t)}$$ general soln.

$Q(0) = 100 = 50 - C \Rightarrow C = -50$

$$\boxed{Q(t) = 50\left(1 + e^{-\frac{1}{20}t}\right)g}$$

5 $$\boxed{\lim_{t\to\infty} Q(t) = 50g}$$

4. (30 points). Find the unique solution to the second-order initial value problem using Laplace transforms:

$$y''(t) + 2y'(t) + 5y(t) = \delta(t-4), \quad y(0) = 1 \text{ and } y'(0) = 0.$$

$$(s^2+2s+5)(\mathcal{L}y)(s) \underbrace{- s - 2}_{\text{from } y(0)=1} = e^{-4s}$$

5 $$(\mathcal{L}y)(s) = \frac{s}{s^2+2s+5} + \frac{2}{s^2+2s+5} + \frac{e^{-4s}}{s^2+2s+5}$$

$$s^2+2s+5 = (s+1)^2 + 2^2 \quad \text{(irreducible)}$$

$$(\mathcal{L}y)(s) = \underbrace{\frac{s+1}{(s+1)^2+2^2}}_{\mathrm{I}} + \frac{1}{2}\underbrace{\frac{2}{(s+1)^2+2^2}}_{\mathrm{II}} + \frac{1}{2} e^{-4s} \cdot \underbrace{\frac{2}{(s+1)^2+2^2}}_{\mathrm{III}}$$

5 I ILT $e^{-t}\cos 2t \quad (b=-1)$

5 II ILT $\frac{1}{2} e^{-t} \sin 2t \quad (b=-1)$

5 III ILT $\frac{1}{2} e^{-4s} H(s) \to \frac{1}{2} u_4(t)\, h(t-4)$

where $(\mathcal{L}h)(s) = H(s)$

$H(s) = \frac{2}{(s+1)^2+2^2}$ so $h(t) = e^{-t}\sin 2t$ as in II

$= \frac{1}{2} u_4(t)\, e^{-(t-4)} \sin 2(t-4)$.

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Summary:

$$\boxed{y(t) = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t + \frac{1}{2}u_4(t)\, e^{-(t-4)}\sin(2(t-4))}$$

Check: $y(0)=1$ & $y'(0) = -1+1 = 0$. ✓

5. (30 points).

Consider the first-order ODE:

$$xy'(x) + y(x) = \cos x, \quad x > 0.$$

a. Find the most general solution to this ODE.

b. Find the unique solution for $x > 0$ satisfying $y(2\pi) = 2$.

a. Normal form: 5 $y' + \frac{1}{x} y = \frac{\cos x}{x}$ , $x > 0$

$p(x) = \frac{1}{x}$

5 $\mu(x) = e^{\int p(x)dx} = e^{\ln x} = x$ .

$(\mu y)' = (xy)' = \mu q = x \cdot \frac{\cos x}{x} = \cos x$

$\mu y(x) = \sin x + C$

10 $\boxed{y(x) = \frac{\sin x}{x} + \frac{C}{x}}$ for $x > 0$ .

b. $y(2\pi) = \frac{\sin 2\pi}{2\pi} + \frac{C}{2\pi} = 2$

10 $C = 4\pi$ as $\sin 2\pi = 0$ .

$$\boxed{y(x) = \frac{\sin x}{x} + \frac{4\pi}{x}}$$

6. (30 points). Consider the following nonhomogeneous, second-order ODE:

$$y''(t) - y'(t) - 6y(t) = e^{-2t}. \qquad g(t) = e^{-2t}$$

a. Find a set of independent solutions for the associated homogeneous ODE. Make sure you verify that the two solutions are independent.

b. Find a particular solution to the nonhomogeneous ODE.

c. Write the general solution to the ODE.

a. $r^2 - r - 6 = 0 = (r-3)(r+2)$ 5 $\boxed{\begin{matrix} y_1(t) = e^{3t} & y_2(t) = e^{-2t} \\ y_1'(t) = 3e^{3t} & y_2'(t) = -2e^{-2t} \end{matrix}}$

Wronskian:

$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)\,y_2(t) = -2e^{t} - (3e^{t})$

5 $= -5e^{t} \neq 0$ never zero so $y_1$ & $y_2$ are independent.

b. Variation of Parameters: $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$

5 $u_1'(t) = \dfrac{-y_2(t)g(t)}{W(t)} = -\dfrac{e^{-2t} \cdot e^{-2t}}{-5e^{t}} = \dfrac{1}{5}e^{-5t}$

$\boxed{u_1(t) = -\frac{1}{25}e^{-5t}}$

5 $u_2'(t) = \dfrac{y_1(t)g(t)}{W(t)} = \dfrac{e^{3t}e^{-2t}}{-5e^{t}} = -\dfrac{1}{5} \Rightarrow \boxed{u_2(t) = -\frac{1}{5}t}$

$y_p(t) = \underbrace{-\frac{1}{25}e^{-2t}}_{\text{soln to homog ODE}} - \frac{1}{5}te^{-2t}$ $\boxed{y_p(t) = -\frac{1}{5}te^{-2t}}$

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c. 5 $\boxed{y_g(t) = C_1e^{3t} + C_2e^{-2t} - \frac{1}{5}te^{-2t}}$