Quiz 3 11 Feb. 2011

NAME: Solutions

1. (4 points) Find the general solution to y' – (1/x) y = x e^{-x}.

What is the unique solution if y(1) = 0.

2. (6 points) Polluted water flows into a 100 liter tank at a rate of 1 l/sec. It contains a concentration of 4 gms/l of lead. Water leaves the tank at the same rate. If there is initially 200 gms of lead in the tank, how much lead is in the tank at any time? What is the maximum amount of lead that can collect in the tank?

1. $y' - \frac{1}{x}y = xe^{-x}$ $\quad p(x) = -\frac{1}{x}$ $\quad \int p(x)dx = -\ln x$ $\quad \mu(x) = e^{\int p(x)dx} = e^{-\ln x}$

$= e^{\ln x^{-1}} = \frac{1}{x}$.

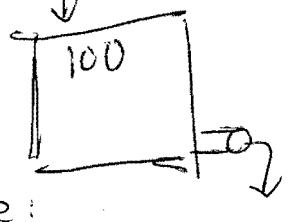
$\mu(x) = \frac{1}{x}$. $\quad q(x) = xe^{-x}$

$(\mu y)' = \mu q \Rightarrow \mu y = \int \frac{1}{x} \cdot xe^{-x}\,dx = \int e^{-x}\,dx = -e^{-x} + C$

General Soln. $\boxed{y(x) = -xe^{-x} + Cx}$ $\quad y(1) = 0 = -e^{-1} + C \Rightarrow C = \frac{1}{e}$

Unique Soln $\boxed{y(x) = -xe^{-x} + xe^{-1}}$

2. 4gm/l, 1l/sec $\quad Q(t)$ amount of lead in tank $\quad Q(0) = 200$ gms



$\frac{dQ}{dt} = [\text{flow in}] - [\text{flow out}]$

$= 4\frac{\text{gm}}{\text{l}} \cdot 1\frac{\text{l}}{\text{sec}} - \frac{Q(t)}{100}\frac{\text{gm}}{\text{l}} \cdot 1\frac{\text{l}}{\text{sec}}$

$= 4 - \frac{1}{100}Q(t)$ gm/sec.

Solve: $\frac{dQ}{dt} = 4 - \frac{1}{100}Q$

Separate $\frac{dQ}{4 - \frac{1}{100}Q} = dt \Rightarrow -100\ln\left|4 - \frac{1}{100}Q\right| = t + C$

$\ln\left|4 - \frac{1}{100}Q\right| = \frac{-1}{100}t + C$

$4 - \frac{1}{100}Q = Ce^{-t/100}$

$\boxed{Q(t) = 400 + Ce^{-t/100}}$ general soln.

Initial Cond. $Q(0) = 400 + C = 200 \Rightarrow C = -200$

$\boxed{Q(t) = 200(2 - e^{-t/100})}$

$\boxed{\lim_{t\to\infty} Q(t) = 400 \text{ gms}}$ Max. amount