MA214-002 Spring 2011
Test #1 100 Points
18 February 2011

INSTRUCTIONS: PLEASE WORK ALL FIVE PROBLEMS BELOW. PLEASE WRITE YOUR NAME ON EACH PAGE. NO CALCULATORS, BOOKS, PAPERS, OR NOTES ARE ALLOWED.

NAME: Solutions

1. (20 points).

Find the most general and then the unique solution to the ODE with the given initial condition:

$$y^2(1+x^3)y'(x) = 6x^2, \quad y(0) = 0.$$

Separate variables: $y^2 dy = \frac{6x^2}{1+x^3} dx$ $u = 1+x^3$, $du = 3x^2 dx$

Integrate: $\frac{1}{3}y^3 + C_1 = 2\ln|1+x^3| + C_2$

$y^3 = 6\ln|1+x^3| + C$

$\boxed{y(x) = \sqrt[3]{6\ln|1+x^3| + C}}$ General Soln.

Initial Cond:

$y(0) = \sqrt[3]{\ln 1 + C} = \sqrt[3]{C} = 0$

$C = 0$.

$\boxed{y(x) = \sqrt[3]{6\ln|1+x^3|}}$ unique Soln.

Check: $y' = \frac{1}{3}(6\ln|1+x^3|)^{-2/3} \cdot \frac{6}{1+x^3} \cdot 3x^2$

$= \frac{6x^2}{1+x^3} \cdot \frac{1}{y^2}$

or $y^2(1+x^3)y' = 6x^2$. ✓

2. (20 points). The population of squirrels on campus begins with an initial population of 100. The population increases at a rate of 5% per year so $r = 1/20$. The campus hawks eat 10 squirrels per year so $k = -10$. Assume the growth and eating are done continuously throughout the year. How long does it take for the squirrel population to go to zero? The ODE satisfied by the population $P(t)$ at time $t \geq 0$ is

$$P'(t) = rP(t) + k.$$

First Solve : $\frac{dP}{rP+k} = dt \qquad \int \frac{dP}{rP+k} = \frac{1}{r}\ln|rP+k| = t+c$

$\Rightarrow \ln(rP+k) = rt + c \Rightarrow \boxed{P = Ce^{rt} - k/r}$

Initial Cond: $P(t=0) = P_0 = C - k/r \Rightarrow C = P_0 + k/r$

$$\boxed{P(t) = (P_0 + k/r)e^{rt} - k/r \;,\; P(t=0) = P_0}$$

Now use the values: $r = \frac{1}{20}$, $k = -10$, $P_0 = 100$

$P(t) = (100 - 200)e^{+\frac{1}{20}t} + 200$

$\boxed{P(t) = 200 - 100\, e^{\frac{1}{20}t}}$ unique soln.

$P(t=0) = 200 - 100 = 100$ ✓

When $P(t_D) = 0$, t_D satisfies:

$0 = 200 - 100\, e^{rt_D}$ or $2 = e^{\frac{1}{20}t_D}$

$$\boxed{t_D = 20 \ln 2}$$

3. (20 points). Find the unique solution to the initial value problem:

$$(1+t)y' + y = 3t^2, \quad y(0) = 2,$$

for $t \geq 0$.

Integrating factor:

$$y' + \underbrace{\frac{1}{1+t}}_{p(t)} y = \underbrace{\frac{3t^2}{1+t}}_{q(t)}$$

$$\int p(t)\,dt = \ln(1+t) \quad \text{so} \quad \mu(t) = e^{\ln(1+t)} = 1+t$$

(note $t \geq 0$)

$$\frac{d}{dt}(\mu y) = \mu q = 3t^2 \Rightarrow \mu y = \int 3t^2\,dt = t^3 + C$$

$$\boxed{y(t) = \frac{t^3 + C}{1+t}}$$ general soln.

Unique soln satisfies $y(0) = C = 2$

$$\boxed{y(t) = \frac{t^3 + 2}{1+t}}$$

Check:

$$y'(t) = \frac{3t^2}{1+t} - \frac{t^3+2}{(1+t)^2} = \frac{3t^2(1+t) - t^3 - 2}{(1+t)^2} = \frac{3t^2 + 3t^3 - t^3 - 2}{1+t}$$

$$= \frac{3t^2}{1+t} - \left(\frac{t^3+2}{1+t}\right)\frac{1}{1+t} = \frac{3t^2}{1+t} - \frac{1}{1+t}y$$

So

$$(1+t)y' + y = 3t^2 \quad \checkmark$$

4. (20 points). A 400 liter holding pond initially contains 200 liters of pure water. A polluted stream dumps 2 liter per minute of polluted water into the pond. The polluted water has a concentration of 2 grams per liter of pollutant. Water flows out of the pond **more slowly** at 1 liter per minute.

a. How many minutes does it take until the pond overflows?

b. What is the volume of water in the pond at any time $t \geq 0$?

c. Let $Q(t)$ be the amount of pollutant, in grams, in the pond at time $t \geq 0$. Find $Q(t)$. Remember:

$$\frac{dQ}{dt} = [\text{flow in}] - [\text{flow out}]\,.$$

d. How much pollutant is in the pond when it overflows?

a) $2\,l/\text{min} - 1\,l/m = 1\,l/m$ net gain. $400-200\,l$ means we need $200\,l$ before overflowing so at $1\,l/m$ we wait $\boxed{200 \text{ min}}$

b.) The volume at time $t \geq 0$ is. $V(t) = (200+t)\,l$ since 1 liter is added each min.

c.) $$\frac{dQ}{dt} = 2\frac{l}{\text{min}} \cdot 2\frac{g}{l} - \frac{Q(t)}{200+t}\frac{g}{l}\; 1\frac{l}{m} \quad \text{all in } \frac{g}{m}.$$

Note that the concentration in the tank at time t is $Q(t)/(200+t)$ g/l.

$$\frac{dQ}{dt} = 4 - \frac{Q}{200+t} \left(\frac{g}{m}\right) \quad \& \quad Q(t=0) = 0 \quad (\text{pure water})$$

$$\Rightarrow \quad Q' + \frac{1}{200+t} Q = 4 \qquad \mu = \exp\left(\int \frac{dt}{200+t}\right) = 200+t$$

$$[(200+t)Q]' = \int 4(200+t)\,dt = 800t + 2t^2 + C$$

$Q(t=0)=0 \Rightarrow C=0$.

General Soln. $$\boxed{Q(t) = \frac{800t + 2t^2 + C}{200+t}}$$

Unique Soln. $$\boxed{Q(t) = \frac{800t + 2t^2}{200+t}.}$$ At $t=200$:

d) $$Q(200) = \frac{8\times10^2 \times 2\times10^2 + 2\times4\times10^4}{4\times10^2} = \frac{24\times10^4}{4\times10^2} = \underline{600}$$

$\boxed{600 \text{ gms}}$

5. (20 points). The velocity of a body of mass $m > 0$ falling in a uniform gravitational field with constant $g > 0$ and experiencing air resistance $\gamma > 0$ satisfies the ODE

$$m\frac{dv}{dt} = -mg - \gamma v, \quad v(t=0) = 0.$$

a. Find the unique solution to the ODE.

b. What is the terminal velocity?

c. How long does it take to reach one-half of the terminal velocity?

a) $\frac{dV}{dt} = -\left[g + \frac{\gamma}{m}V\right] \Rightarrow \frac{dV}{g + \frac{\gamma}{m}V} = -dt$

$\Rightarrow \frac{m}{\gamma}\ln\left|g + \frac{\gamma}{m}V\right| = -t + c$ general soln.

$g + \frac{\gamma}{m}V = Ce^{-\frac{\gamma}{m}t}$ or $\boxed{V(t) = Ce^{-\frac{\gamma}{m}t} - \frac{gm}{\gamma}}$

Initial Condition: $V(0) = 0 = C - \frac{gm}{\gamma}$ so $C = \frac{gm}{\gamma}$

$\boxed{V(t) = -\frac{gm}{\gamma}\left[1 - e^{-\frac{\gamma}{m}t}\right]}$ unique soln.

b) Terminal Velocity $\lim_{t\to\infty} V(t) = -\frac{gm}{\gamma}$ (negative since it is directed downward.

$\boxed{V_{term} = -\frac{gm}{\gamma}}$

c.) Find $T > 0$ so $V(T) = -\frac{1}{2}\frac{gm}{\gamma} = -\frac{gm}{\gamma}\left[1 - e^{-\frac{\gamma T}{m}}\right]$

(the minus sign is essential!) or

$-\frac{1}{2} = -1 + e^{-\gamma T/m}$

$\frac{1}{2} = e^{-\gamma T/m}$

take logs: $\underbrace{\ln\frac{1}{2} = -\ln 2}_{\text{You should know this}} = -\frac{\gamma T}{m} \Rightarrow \boxed{T = \frac{m}{\gamma}\ln 2}$

2 neg signs! so T is positive