MA214 002 Spring 2011
Test #2 - 100 points
8 April 2011

NAME: Solutions

INSTRUCTIONS: PLEASE WORK ALL **FIVE** PROBLEMS BELOW. PLEASE WRITE YOUR NAME ON EACH PAGE. NO CALCULATORS, BOOKS, PAPERS, OR NOTES ARE ALLOWED.
Each problem is worth 20 points.

1. Consider the homogeneous ODE:

$$y''(t) + 4y'(t) + 6y(t) = 0.$$

   (a) Find a pair of linearly independent **real** solutions to the ODE. Be sure to check that they are independent.

   (b) Write the most general real solution to the ODE.

a.) $r^2 + 4r + 6 = 0$ roots $(-4 \pm [16 - 4 \cdot 6]^{\frac{1}{2}})/2 = -2 \pm \sqrt{2}\, i$ $(\sqrt{8} = 2\sqrt{2})$

2 real solns. $y_1(t) = e^{-2t} \cos \sqrt{2}t$ $\quad y_2(t) = e^{-2t} \sin \sqrt{2}t$

Wronskian: $y_1'(t) = -2e^{-2t} \cos \sqrt{2}t - \sqrt{2} e^{-2t} \sin \sqrt{2}t$

$y_2'(t) = -2e^{-2t} \sin \sqrt{2}t + \sqrt{2} e^{-2t} \cos \sqrt{2}t$

$$W(y_1, y_2)(t) = y_1(t) y_2'(t) - y_1'(t) y_2(t)$$
$$= e^{-4t}\left(-2 \cos \sqrt{2}t \sin \sqrt{2}t + \sqrt{2} \cos^2 \sqrt{2}t\right)$$
$$- e^{-4t}\left(-2 \cos \sqrt{2}t \sin \sqrt{2}t - \sqrt{2} \sin^2 \sqrt{2}t\right)$$

$\boxed{W(y_1, y_2)(t) = \sqrt{2}\, e^{-4t}}$ Since this never vanishes, $\{y_1, y_2\}$ are independent.

b.) $y(t) = C_1 e^{-2t} \cos \sqrt{2}t + C_2 e^{-2t} \sin \sqrt{2}t$.

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2. Consider the ODE:

$$y'' + 2y' + y = \cos t.$$

(a) Using an intelligent guess, find the most general real solution to the ODE.

(b) Find the unique real solution satisfying the initial conditions:

$$y(0) = 0, \quad y'(0) = 0.$$

a.) Homog. ODE: $y'' + 2y' + y = 0$. $r^2 + 2r + 1 = (r+1)^2 = 0$

$y_1(t) = e^{-t}$, $y_2(t) = te^{-t}$

so $\boxed{y_h(t) = C_1 e^{-t} + C_2 t e^{-t}}$

Particular Soln.

since $g(t) = \cos t$ is independent of $\{y_1, y_2\}$

$y_p(t) = A\cos t + B\sin t$

$y_p'(t) = -A\sin t + B\cos t$

$y_p''(t) = -A\cos t - B\sin t$

$y_p'' + 2y_p' + y_p = (\cos t)(-A + 2B + A) + (\sin t)(-B - 2A + B)$

$= \cos t \Rightarrow B = \frac{1}{2} \quad A = 0$

$\boxed{y_p(t) = \frac{1}{2}\sin t}$ $\boxed{y_g(t) = C_1 e^{-t} + C_2 t e^{-t} + \frac{1}{2}\sin t}$

b.) $y(0) = C_1 = 0$

$y'(t) = C_2 e^{-t} - C_2 t e^{-t} + \frac{1}{2}\cos t$

$y'(0) = C_2 + \frac{1}{2} = 0 \quad C_2 = -\frac{1}{2}$

$\boxed{y(t) = -\frac{1}{2} t e^{-t} + \frac{1}{2}\sin t}$

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4. Find the most general solution to the ODE using variation of parameters:

$$y''(t) + y'(t) - 2y(t) = te^t.$$

$$r^2 + r - 2 = (r+2)(r-1) = 0$$

$$y_1 = e^{-2t} \quad y_2 = e^t$$

$$y_1' = -2e^{-2t} \quad y_2' = e^t$$

$$W = e^{-t} - (-2e^{-t}) = 3e^{-t} \neq 0$$

$$u_1' = -\frac{y_2 g}{W} = -\frac{e^t \, te^t}{3e^{-t}} = -\frac{1}{3}te^{3t}$$

$$u_2' = \frac{y_1 g}{W} = \frac{e^{-2t} \, te^t}{3e^{-t}} = \frac{1}{3}t$$

$$\int te^{3t}dt = \frac{1}{3}te^{3t} - \int \frac{1}{3}e^{3t}dt = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t}$$

$u = t \quad du = dt$

$dv = e^{3t}dt \quad v = \frac{1}{3}e^{3t}$

$$u_1(t) = -\frac{1}{3}\int te^{3t}dt = -\frac{1}{9}te^{3t} + \frac{1}{27}e^{3t}$$

$$u_2(t) = \frac{1}{6}t^2$$

part of y_h

$$y_p(t) = -\frac{1}{9}te^{t} + \frac{1}{27}e^t + \frac{1}{6}t^2e^t$$

$$\boxed{y_g(t) = C_1e^{-2t} + C_2e^t + \left(-\frac{1}{9}\right)te^t + \frac{1}{6}t^2e^t}$$

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5. Consider the first-order ODE:

$$y' = 3t^2, \; y(0) = 0.$$

a. Find unique solution of this initial value problem.

b. For the ODE $y'(t) = f(t, y)$, use the Euler method to write y_{n+1} in terms of the uniform step size h, and the initial condition (t_0, y_0).

c. Apply the Euler method to the ODE above $y' = 3t^2$ with step size $h = 1$ and initial condition $(0, 0)$. What is the formula for y_{n+1}? Make a table with three columns labeled by t_j, y_j, and $y(t_j)$ (the exact solution) for $j = 0, 1, 2, 3$. Compare the Euler values y_j with the exact solution $y(t_j)$.

a.) $y(t) = t^3$ by integration

b.) $y_{n+1} = y_n + h\, f(t_n, y_n)$ $\quad n = 0, 1, 2, \ldots$

c.) $h = 1 \quad f(t,y) = 3t^2$ (independent of y!).

$t_0 = 0 \qquad y_0 = 0 \qquad\qquad y(0) = 0$

$t_1 = 1 \qquad y_1 = y_0 + f(0,0) = 0 \qquad\qquad y(1) = 1$

$t_2 = 2 \qquad y_2 = y_1 + f(t_1, y_1) = 0 + 3 = 3 \qquad\qquad y(2) = 8$

$t_3 = 3 \qquad y_3 = y_2 + f(t_2, y_2) = 3 + f(2,3) \qquad\qquad y(3) = 27$

$= 3 + 12 = 15$

(0 0)
(1 0)
(2 3)
(3 15)

Compare with $y(t) = t^3$

$|y(t_3) - y_3| = 27 - 15 = 12$ so ~40% off!

This is because $y(t)$ increases quickly and h is too big!

j	t_j	y_j	$y(t_j)$
0	0	0	0
1	1	0	1
2	2	3	8
3	3	15	27