1. Consider the homogeneous ODE:

\[ y''(t) + 4y'(t) + 6y(t) = 0. \]

(a) Find a pair of linearly independent real solutions to the ODE. Be sure to check that they are independent.

(b) Write the most general real solution to the ODE.

1.4-1/2 d. c0. 

\[
\begin{align*}
\text{roots } & \quad (-4 \pm \sqrt{16-4 \cdot 6})/2 = -2 \pm \sqrt{2} \quad (18 = 2\sqrt{2}) \\
\text{2 real solns. } & \\
\text{Wronskian: } & \\
W(y_1, y_2) & = y_1(t) y_2'(t) - y_1'(t) y_2(t) \\
& = e^{-4t} \left( -2 \cos \sqrt{2} t \sin \sqrt{2} t + \sqrt{2} \cos^2 \sqrt{2} t \right) \\
& \quad - e^{-4t} \left( -2 \cos \sqrt{2} t \sin \sqrt{2} t - \sqrt{2} \sin^2 \sqrt{2} t \right) \\
\text{W(y_1, y_2)(t)} & = \sqrt{2} e^{-4t} \\
\text{Since this never vanishes, } \{y_1, y_2\} & \text{ are independent.}
\end{align*}
\]

\[
\begin{align*}
\text{6.) } y_1(t) & = C_1 e^{-2t} \cos \sqrt{2} t + C_2 e^{-2t} \sin \sqrt{2} t \\
\end{align*}
\]
2. Consider the ODE:
\[ y'' + 2y' + y = \cos t. \]

(a) Using an intelligent guess, find the most general real solution to the ODE.

(b) Find the unique real solution satisfying the initial conditions:
\[ y(0) = 0, \quad y'(0) = 0. \]

(a) Homog. ODE:
\[ y'' + 2y' + y = 0 \]
\[ r^2 + 2r + 1 = (r + 1)^2 = 0 \]
\[ y_1(t) = e^{-t}, \quad y_2(t) = te^{-t} \]
so
\[ y_h(t) = C_1 e^{-t} + C_2 te^{-t} \]

Particular Soln.
\[ y_p(t) = A \cos t + B \sin t \]
since \( g(t) = \cos t \)
\[ y_p(t) = -A \sin t + B \cos t \]
\[ y''_p(t) = -A \cos t - B \sin t \]
\[ y'' + 2y' + y_p = (\cos t)(-A + 2B + A) + (\sin t)(-B - 2A + B) = \cos t \]
\[ \Rightarrow \begin{cases} A = 0 \\ B = \frac{1}{2} \end{cases} \]
\[ y_p(t) = \frac{1}{2} \sin t \]
\[ y_g(t) = C_1 e^{-t} + C_2 te^{-t} + \frac{1}{2} \sin t \]

(b) \[ y(0) = C_1 + C_2 = 0 \]
\[ y'(0) = C_2 e^{-t} - C_2 te^{-t} + \frac{1}{2} \cos t \]
\[ y'(0) = C_2 + \frac{1}{2} = 0 \]
\[ C_2 = -\frac{1}{2} \]
\[ y(t) = -\frac{1}{2} te^{-t} + \frac{1}{2} \sin t \]
4. Find the most general solution to the ODE using variation of parameters:

\[ y''(t) + y'(t) - 2y(t) = te^t. \]

\[ \Delta^2 + \Delta - 2 = (\Delta + 2)(\Delta - 1) = 0 \]

\[ y_1 = e^{-t}, \quad y_1' = e^t \]

\[ y_1' = -2e^{-t}, \quad y_1' = e^t \]

\[ W = e^{-t} - l - 2e^{-t} = 3e^{-t} + 0 \]

\[ u_1' = -\frac{y_2g}{W} = -\frac{e^t \cdot te^t}{3e^{-t}} = -\frac{1}{3} \cdot te^{3t} \]

\[ u_2' = \frac{y_1g}{W} = \frac{e^{-2t} \cdot te^t}{3e^{-t}} = \frac{1}{3} t \]

\[ \int te^{3t} dt = \frac{1}{3} te^{3t} - \int \frac{1}{3} e^{3t} dt = \frac{1}{3} te^{3t} - \frac{1}{9} e^{3t} \]

\[ u = t \quad du = dt \]

\[ dv = e^{3t} dt \quad v = \frac{1}{3} e^{3t} \]

\[ u_1(t) = -\frac{1}{3} \int te^{3t} dt = -\frac{1}{9} te^{3t} + \frac{1}{3} \frac{1}{3} e^{3t} \]

\[ u_2(t) = \frac{1}{6} t^2 \]

Partial of \( y_1 \)

\[ y_1(t) = -\frac{1}{9} te^{-t} + \frac{1}{9} e^t + \frac{1}{6} te^t \]

\[ y_2(t) = Ce^{-t} + Ce^t + \left(\frac{1}{9}\right) te^t + \frac{1}{6} t^2 e^t \]

\[ y_2(t) = Ce^{-t} + Ce^t + \left(\frac{1}{9}\right) te^t + \frac{1}{6} t^2 e^t \]
5. Consider the first-order ODE:

\[ y' = 3t^2, \ y(0) = 0. \]

a. Find unique solution of this initial value problem.

b. For the ODE \( y'(t) = f(t,y) \), use the Euler method to write \( y_{n+1} \) in terms of the uniform step size \( h \), and the initial condition \((t_0, y_0)\).

c. Apply the Euler method to the ODE above \( y' = 3t^2 \) with step size \( h = 1 \) and initial condition \((0,0)\). What is the formula for \( y_{n+1} \)? Make a table with three columns labeled by \( t_j \), \( y_j \), and \( y(t_j) \) (the exact solution) for \( j = 0, 1, 2, 3 \). Compare the Euler values \( y_j \) with the exact solution \( y(t_j) \).

\[ f(t,y) = 3t^2 \quad \text{(independent of } y) \]

\[ t_0 = 0, \quad y_0 = 0 \]
\[ t_1 = 1, \quad y_1 = y_0 + f(t_0, y_0) = 0 \]
\[ t_2 = 2, \quad y_2 = y_1 + f(t_1, y_1) = 0 + 3 = 3 \]
\[ t_3 = 3, \quad y_3 = y_2 + f(t_2, y_2) = 3 + f(2, 3) = 3 + 12 = 15 \]

Compare with \( y(t) = t^3 \)

\[ |y(t_3) - y_3| = 27 - 15 = 12 \ \text{so 40\% off!} \]

This is because \( y(t) \) increases quickly and \( h \) is too big!