MA214-003 Fall 2009
Final Exam - 180 Points
16 December 2009, 10:30–12:30 PM, CB 203

INSTRUCTIONS: PLEASE WORK ALL 6 PROBLEMS BELOW. PLEASE WRITE YOUR NAME AND SECTION NUMBER ON EACH PAGE. NO CALCULATORS, BOOKS, PAPERS, OR NOTES ARE ALLOWED.

LAPLACE TRANSFORM FORMULAS ON THE LAST PAGE.

NAME: Solutions

PROBLEM	MAXIMUM GRADE	SCORE
1	30	
2	30	
3	30	
4	30	
5	30	
6	30	
TOTAL	180	

1. (30 points).

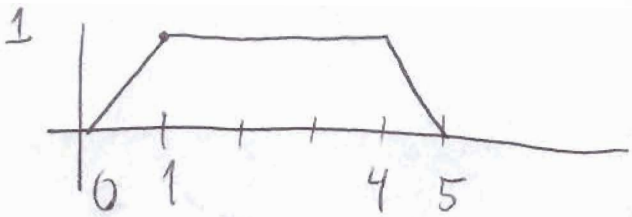
a. Write the function g in terms of the step function $u_c(t)$:

$$g(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t < 4 \\ 5-t & 4 \le t < 5 \\ 0 & t \ge 5 \end{cases}$$

b. Compute the Laplace transform of $g(t)$.

a)

$g(t)$



$$g(t) = t - t u_1(t) + u_1(t) - u_4(t) + (5-t)u_4(t) - (5-t)u_5(t)$$

$$= t + (1-t)u_1(t) + (4-t)u_4(t) + (t-5)u_5(t)$$

) $(\mathcal{L}g)(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-4s}}{s^2} + \frac{e^{-5s}}{s^2} = \frac{1}{s^2}\left(1 - e^{-s} - e^{-4s} + e^{-5s}\right)$

$$\boxed{g(t) = t - (t-1)u_1(t) - (t-4)u_4(t) + (t-5)u_5(t)}$$

$$\boxed{(\mathcal{L}g)(s) = \frac{1}{s^2}\left(1 - e^{-s} - e^{-4s} + e^{-5s}\right)}$$

2. (30 points). Find the unique solution to the second-order initial value problem:

$$y''(t) + 9y(t) = \delta(t-3), \quad y(0) = 1 \text{ and } y'(0) = 1.$$

Take the LT :

$$s^2(\mathcal{L}y)(s) - s - 1 + 9(\mathcal{L}y)(s) = e^{-3s}$$

$$(s^2+9)(\mathcal{L}y)(s) = e^{-3s} + 1 + s$$

$$\boxed{(\mathcal{L}y)(s) = \underset{(i)}{\frac{e^{-3s}}{s^2+9}} + \underset{(ii)}{\frac{1}{s^2+9}} + \underset{(iii)}{\frac{s}{s^2+9}}}$$

ILT:

(i) $e^{-3s}\dfrac{1}{s^2+9} = \dfrac{1}{3}e^{-3s}H(s)$ $\qquad H(s) = \dfrac{3}{s^2+3^2}$

$(\mathcal{L}^{-1}H)(t) = \sin 3t$

$ILT = u_3(t)\cdot\dfrac{1}{3}\sin 3(t-3)$

(ii) $\dfrac{1}{3}\sin 3t$

(iii) $\cos 3t$

So $\boxed{y(t) = \dfrac{1}{3}u_3(t)\sin(3(t-3)) + \dfrac{1}{3}\sin 3t + \cos 3t}$.

Check: $y(0) = 1$ $\quad$ since the u_3 term is zero for $0 \le t < 3$.

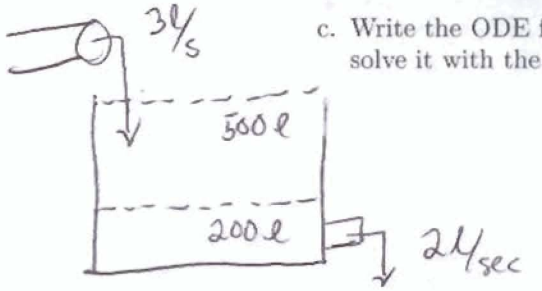
$y'(0) = 1$

3. (30 points). A 500ℓ tank initially contains a 200ℓ solution of $100kg$ of salt dissolved in water. Water containing $1kg/\ell$ of solution enters the tank at a rate of $3\ell/sec$ and leaves at a rate of $2\ell/sec$.

a. How much solution is in the tank at any time $t > 0$?

b. How long until the tank overflows?

c. Write the ODE for the quantity of salt in the tank at any time $t > 0$ and solve it with the initial condition $Q(0) = 100kg$.



Initial Vol. $= 200\ell$

$Q(t) \equiv$ amount of salt in tank at time t in kg

$Q(0) = 100$ kg

a.) Flow in $3\ell/s$ - Flow out $2\ell/s = 1\ell/s$ filling up the tank!

Vol. of solution $= 200\ell + 1(\ell/s)\cdot t$ sec $= \boxed{(200+t)\,\ell}$

b.) If the volume is increasing like $(200+t)\ell$ when $\boxed{t = 300}$ the tank is full.

c.) $$\frac{dQ}{dt} = 1\frac{kg}{\ell}\cdot 3\frac{\ell}{sec} - \frac{Q(t)}{200+t}\frac{kg}{\ell}\cdot 2\frac{\ell}{sec} = 3 - \frac{2Q}{200+t}\ \frac{kg}{sec}$$

$$\boxed{\frac{dQ}{dt} + \frac{2Q}{200+t} = 3}$$

Integrating factor $p(t) = \frac{2}{200+t}$ $\quad \int p(t)\,dt = 2\ln(200+t)$

$$\mu = (200+t)^2 = e^{2\ln(200+t)}$$

$$(Q\mu)' = (\mu q) = 3(200+t)^2 \Rightarrow \mu Q = 3\int (200+t)^2 dt$$

$$= (200+t)^3 + C$$

$$\boxed{Q(t) = (200+t) + C(200+t)^{-2}}$$

check: $Q'(t) = 1 - \frac{2C}{(200+t)^3}$ subst.

$+$

$\frac{2Q}{200+t} = 2 + \frac{2C}{(200+t)^3}$ $\Big\} = 3$ ✓

$$Q(0) = 100 = 200 + \frac{C}{(200)^2}$$

$$-10^2 = \frac{C}{4\times10^4} \Rightarrow C = -4\times10^6$$

$$\boxed{Q(t) = (200+t) - \frac{4\times10^6}{(200+t)^2}}$$

$Q(0) = 200 - \frac{4\times10^6}{4\times10^4} = 200 - 100$

$= 100$ ✓

4. (30 points). Consider the following nonhomogeneous, second-order initial value problem:

$$y''(t) + 2y'(t) + 5y(t) = t, \quad y(0) = 1, \; y'(0) = 0.$$

a. Find a set of independent solutions for the associated homogeneous ODE. Make sure you verify that the two solutions are independent.

b. Find a particular solution to the nonhomogeneous ODE.

c. Write the general solution to the ODE.

d. Write the unique solution to the initial value problem.

a.) $y'' + 2y' + 5y = 0 \quad r^2 + 2r + 5 = 0 \quad \text{roots} = \dfrac{-2 \pm [4-20]^{\frac{1}{2}}}{2} = -1 \pm 2i$

Euler $\Rightarrow$ $\boxed{y_1(t) = e^{-t}\cos 2t \qquad y_2(t) = e^{-t}\sin 2t}$

$y_1'(t) = e^{-t}(-\cos 2t - 2\sin 2t) \qquad y_2'(t) = e^{-t}(-\sin 2t + 2\cos 2t)$

$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \boxed{2e^{-2t}} \neq 0$ so independent

$y_h(t) = C_1 e^{-t}\cos 2t + C_2 e^{-t}\sin 2t$

b.) Guess $y_p(t) = At + B$ since t is indep of y_1 & y_2.

$y_p'(t) = A \qquad 5At = t \quad A = \frac{1}{5}$

$y_p''(t) = 0 \qquad \frac{2}{5} + 5B = 0 \quad B = -\frac{2}{25}$

Substitute: $2A + 5At + 5B = t$

$$\boxed{y_p(t) = \frac{1}{5}t - \frac{2}{25}}$$

c.) $y_{gen}(t) = y_h(t) + y_p(t) = C_1 e^{-t}\cos 2t + C_2 e^{-t}\sin 2t + \frac{1}{5}t - \frac{2}{25}$

$y'(t) = -C_1 e^{-t}(\cos 2t + 2\sin 2t) + C_2 e^{-t}(\sin 2t + 2\cos 2t) + \frac{1}{5}$

d) $y(0) = C_1 - \frac{2}{25} = 1 \qquad C_1 = \frac{27}{25}$

$y'(0) = -C_1 + 2C_2 + \frac{1}{5} = 0 \qquad 2C_2 = -\frac{1}{5} + \frac{27}{25} = \frac{22}{25}$

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$C_2 = \frac{11}{25}$

$$\boxed{y(t) = \frac{27}{25}e^{-t}\cos 2t + \frac{11}{25}e^{-t}\sin 2t + \frac{1}{5}t - \frac{2}{25}}$$

5. (30 points).

a. Find the unique solution to the initial value problem:

$$\frac{dy}{dt} = y(3-y), \quad y(0) = 1.$$

b. Compute $\lim_{t\to\infty} y(t)$.

a. $\int \frac{dy}{y(3-y)} = \int dt$ Partial fraction $\frac{1}{y(3-y)} = \left(\frac{1}{y} + \frac{1}{3-y}\right)\left(\frac{1}{3}\right)$

$\frac{1}{3}\left[\ln y - \ln(3-y)\right] = \frac{1}{3}\ln\left(\frac{y}{3-y}\right) = t + c$

$\frac{y}{3-y} = Ce^{3t}$ solve for y: $y(1+Ce^{3t}) = 3Ce^{3t}$

$$\boxed{y(t) = \frac{3Ce^{3t}}{1+Ce^{3t}}}$$

Initial Condition: $y(0) = 1 = \frac{3C}{1+C} \Rightarrow 1 + C = 3C$
$C = \frac{1}{2}$

$$\boxed{y(t) = \frac{3e^{3t}}{2+e^{3t}}}, \quad y(0) = \frac{3}{2+1} = 1 \checkmark$$

b. $\lim_{t\to+\infty} y(t) = \frac{3}{2e^{-3t}+1} = 3$

$\boxed{\lim_{t\to\infty} y(t) = 3}$ this is a Logistic ODE.

6. (30 points). Consider the nonhomogeneous second order ODE:

$$y''(x) - \frac{3}{x}y'(x) + \frac{4}{x^2}y(x) = \ln x, \quad x > 0.$$

You are told that $y_1(x) = x^2$ and $y_2(x) = x^2 \ln x$ are two solutions to the associated homogeneous ODE.

a. Compute Wronskian of these two solutions. Are they independent for $x > 0$?

b. Find a particular solution to the nonhomogeneous ODE using the variation of parameters method. Integration is done with the substitution $u = \ln x$.

c. Write the most general solution to the nonhomogeneous ODE.

a. $y_1(x) = x^2$ $\quad$ $y_2(x) = x^2 \ln x$

$y_1'(x) = 2x$ $\quad$ $y_2'(x) = 2x\ln x + x$

$W(y_1, y_2)(x) = 2x^3 \ln x + x^3 - 2x^3 \ln x = x^3 > 0$ for $x > 0$ so indep.

b. Variation of Parameters.

$$u_1'(x) = \frac{-y_2(x)g(x)}{W(x)} = -\frac{x^2 \ln x \cdot \ln x}{x^3} = -\frac{(\ln x)^2}{x}$$

$$u_1(x) = -\int \frac{(\ln x)^2}{x}\,dx = -\int u^2\,du = -\frac{1}{3}u^3 = -\frac{1}{3}(\ln x)^3.$$

$u = \ln x$

$du = \frac{dx}{x}$

$$u_2'(x) = \frac{y_1(x)g(x)}{W(x)} = \frac{x^2 \ln x}{x^3} = \frac{\ln x}{x} \Rightarrow u_2(x) = \int \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2$$

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = -\frac{1}{3}x^2(\ln x)^3 + \frac{1}{2}x^2(\ln x)^3 = \frac{1}{6}x^2(\ln x)^3$$

$$\boxed{y_p(x) = \frac{1}{6}x^2(\ln x)^3}$$

c. General soln:

$$\boxed{y(x) = C_1 x^2 + C_2 x^2 \ln x + \frac{1}{6}x^2(\ln x)^3}$$

Some Formulas

1. **Summation formulas:**

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

2. **Areas:**

   (a) Triangle $A = \frac{bh}{2}$

   (b) Circle $A = \pi r^2$

   (c) Rectangle $A = lw$

   (d) Trapezoid $A = \frac{b_1 + b_2}{2} h$

3. **Volumes:**

   (a) Rectangular Solid $V = lwh$

   (b) Sphere $V = \frac{4}{3}\pi r^3$

   (c) Cylinder $V = \pi r^2 h$

   (d) Cone $V = \frac{1}{3}\pi r^2 h$