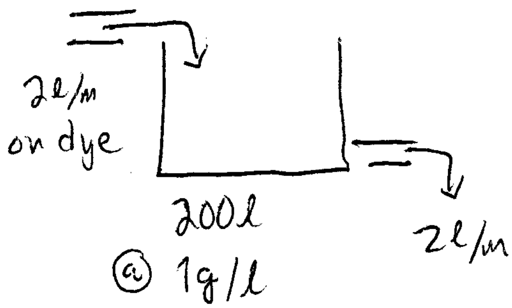
MA214

Problem Set 3

pg 59 #1.



Find T so that concentration is 1% of 1 g/l = 10^{-2} g/l

$Q(t) \equiv$ # grams of dye in tank at time t.

$$\frac{dQ}{dt} = \begin{pmatrix}\text{no dye}\\ \text{coming}\\ \text{in}\end{pmatrix} - \frac{Q(t)}{200}\left(\frac{g}{l}\right) \times 2\,\frac{l}{min} \qquad \text{units } \frac{g}{min}$$

$Q(t=0) = 1\,g/l \times 200\,l = 200\,g$.

Solve $\frac{dQ}{dt} = -\frac{Q}{100}$ or $Q(t) = 200e^{-t/100}$ grams.

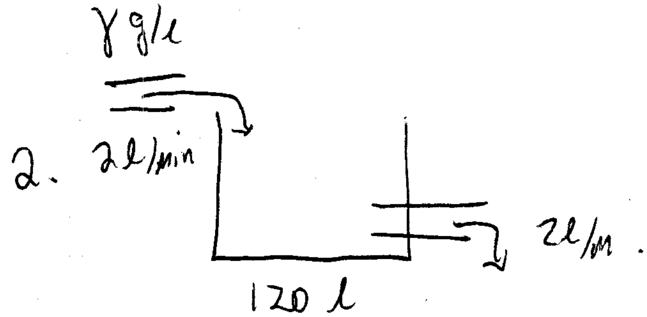
$$Q(T) = 10^{-2}\,\frac{g}{l}\;200\,l = 2\text{ gms.} = 200\,e^{-T/100}$$

$$10^{-2} = e^{-T\,10^{-2}}$$

$$-2\log 10 = -10^{-2}T$$

$$T = 200\log 10 \text{ min.}$$

2\.



$Q(t) \equiv$ # grams salt in tank at time t.

$$\frac{dQ}{dt} = (in) - (out) = 2\gamma - \frac{Q(t)}{120}\cdot 2 \quad \left(\frac{g}{m}\right)$$

$$= 2\gamma - \frac{Q(t)}{60}$$

$Q(t) = C_0 e^{-t/60} + 120\gamma$

Initial Cond. $Q(0) = 0 \Rightarrow C_0 = -120\gamma$

$Q(t) = 120\gamma(1 - e^{-t/60})$ grams

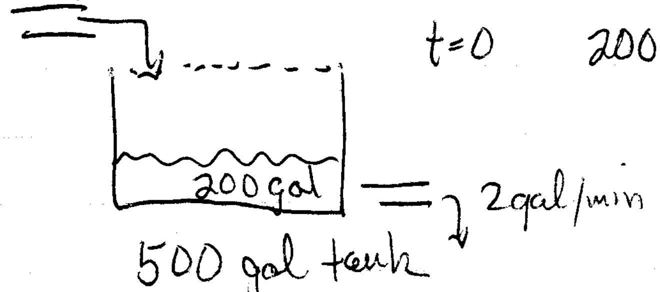
as $t \to \infty$ $\quad Q(t) \to 120\gamma$ equilibrium soln.

1 lb/gal
3 gal/min

(3.2)

t=0 200 gal w/ 100 lb salt

4.



the liquid in the tank increases at 1 gal/min so takes 300 min to reach "overflow level"

Volume at time t is 200 gal + $1\frac{gal}{min}\cdot t$ min

$= (200+t)$ gal.

$$\frac{dQ}{dt} = 1\left(\frac{lb}{gal}\right)\times 3\left(\frac{gal}{min}\right) - \frac{Q(t)}{200+t}\left(\frac{lbs}{gal}\right)\cdot 2\left(\frac{gal}{min}\right)$$

$$\begin{cases} \frac{dQ}{dt} = 3 - \frac{2Q}{200+t} \quad \text{or} \quad \frac{dQ}{dt} + \left(\frac{2}{200+t}\right)Q = 3. \\ Q(t=0) = 100 \end{cases}$$

$p(t) = 2/(200+t)$ so $\mu = (200+t)^2$

$(\mu Q)' = 3\mu = 3(200+t)^2 \Rightarrow \mu Q = (200+t)^3 + C$

$Q(t) = (200+t) + C(200+t)^{-2}$

$Q(0) = 100 = 200 + C/4\times10^4 \Rightarrow C = -4\times10^6$

(i) $Q(t) = (200+t) - \frac{4\times10^6}{(200+t)^2}$ amount in tank

$Q(300) = 500 - \frac{4\times10^6}{25\times10^4} = 484$ lbs.

(ii) Concentration at 300 min = 484 lbs/500 gal = 0.968 lbs/gal

the initial concentration is 0.5 lbs/gal.

(iii) If the tank has infinite capacity

the concentration is $= \frac{Q(t)}{200+t}$ $\left(\frac{200+t}{200+t} - \frac{4\times10^6}{(200+t)^3}\right) \xrightarrow[t\to\infty]{} 1$ lb/gal pg. 95

2. $(2x+4y)+(2x-2y)y'=0$

$\frac{\partial M}{\partial y}=4 \quad \frac{\partial N}{\partial x}=2$ Not exact.

4 $(2xy^2+2y)+(2x^2y+2x)y'=0$

$\frac{\partial M}{\partial y}=4xy+2 \quad \frac{\partial N}{\partial x}=4xy+2$ so Exact.

$\frac{\partial \psi}{\partial x}=2xy^2+2y \Rightarrow \psi(x,y)=x^2y^2+2xy+h(y)$

$\frac{\partial \psi}{\partial y}=2x^2y+2x \Rightarrow \psi(x,y)=x^2y^2+2xy+g(x)$

Compare these 2: take $h=g=0$.

Soln $\psi(x,y)=x^2y^2+2xy=C$.

7. $(e^x\sin y-2y\sin x)dx+(e^x\cos y+2\cos x)dy=0$

$\underbrace{(e^x\sin y-2y\sin x)}_{M}+\underbrace{(e^x\cos y+2\cos x)}_{N}\frac{dy}{dx}=0$

$\frac{\partial M}{\partial y}=e^x\cos y-2\sin x \quad \frac{\partial N}{\partial x}=e^x\cos y-2\sin x$ Exact

$\frac{\partial \psi}{\partial x}=e^x\sin y-2y\sin x \Rightarrow \psi(x,y)=+e^x\sin y+2y\cos x+h(y)$

$\frac{\partial \psi}{\partial y}=e^x\cos y+2\cos x \Rightarrow \psi(x,y)=e^x\sin y+2y\cos x+g(x)$

take $h=g=0$ so $e^x\sin y+2y\cos x=C$ is the soln.

13. $(2x-y)+(2y-x)\frac{dy}{dx}=0$.

$\frac{\partial M}{\partial y}=-1 \quad \frac{\partial N}{\partial x}=-1$

$\psi(x,y)=x^2-yx+h(y)=y^2-xy+g(x)$ take $h(y)=y^2$, $g(x)=x^2$

so $\psi(x,y)=x^2-yx+y^2=C$

(3-4)

$y(1)=3$: $1-3+9=C$ $C=7$

Solve:

$y^2 - yx + x^2 - 7 = 0$

$y = x \pm [x^2 - 4(x^2 - 7)]^{1/2}/2$

$= \frac{1}{2}\{x \pm [28-3x^2]^{1/2}\}$

Valid if $|x| < (28/3)^{1/2}$

18. $M(x) + N(y)y' = 0$

To be exact we need $\frac{\partial M(x)}{\partial y} = 0 = \frac{\partial N(y)}{\partial x}$ ✓

so ok.

20. $\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right) + \left(\frac{\cos y + 2e^{-x}\cos x}{y}\right)\frac{dy}{dx} = 0$

Not exact: $\frac{\partial M}{\partial y} = \frac{-\sin y}{y^2} + \frac{\cos y}{y}$

$\frac{\partial N}{\partial x} = 2\frac{1}{y}e^{-x}(-\cos x - \sin x)$

Multiply by ye^x:

$\tilde{M} = e^x \sin y - 2y\sin x$ $\qquad \frac{\partial \tilde{M}}{\partial y} = e^x\cos y - 2\sin x$ ✓

$\tilde{N} = e^x\cos y + 2\cos x$ $\qquad \frac{\partial \tilde{N}}{\partial x} = e^x\cos y - 2\cos x$

$\psi(x,y) = e^x\sin y + 2y\cos x + h(y)$

$= e^x\sin y + 2y\cos x + g(x)$

} we can take $h = 0 = g$

Soln $\psi(x,y) = e^x\sin y + 2y\cos x = C.$