MA214-03 Problem Set 4 Solutions

3.1.2. $y'' + 3y' + 2y = 0 \qquad r^2 + 3r + 2 = 0 = (r+2)(r+1)$

2 roots $r_1 = -2 \quad r_2 = -1$ Gen. soln $y(t) = C_1 e^{-2t} + C_2 e^{-t}$

4. $2y'' - 3y' + y = 0 \qquad 2r^2 - 3r + 1 = 0 = (2r-1)(r-1)$

2 roots $r_1 = \frac{1}{2} \quad r_2 = 1$ Gen Soln. $y(t) = C_1 e^{t/2} + C_2 e^{t}$

6. $4y'' - 9y = 0. \qquad 4r^2 = 9 \quad r = \pm 3/2$

$$y(t) = C_1 e^{3/2 t} + C_2 e^{-3/2 t}$$

8. $y'' - 2y' - 2y = 0 \qquad r^2 - 2r - 2 = 0 = (r - r_1)(r - r_2)$

$$r_{1,2} = \frac{2 \pm [4+8]^{1/2}}{2} = 1 \pm \sqrt{3}$$

$$y(t) = C_1 e^{(1+\sqrt{3})t} + C_2 e^{(1-\sqrt{3})t}$$

10. $y'' + 4y' + 3y = 0 \qquad r^2 + 4r + 3 = 0 = (r+3)(r+1)$

$y(t) = C_1 e^{-3t} + C_2 e^{-t} \qquad y'(t) = -3C_1 e^{-3t} - C_2 e^{-t}$

$y(0) = 2 \qquad y(0) = C_1 + C_2 = 2$

$y'(0) = -1 \qquad y'(0) = -3C_1 - C_2 = -1$ } add $\Rightarrow -2C_1 = 1$

$C_1 = -\frac{1}{2}$

$C_2 = 5/2$

$y(t) = -\frac{1}{2} e^{-3t} + 5/2\, e^{-t}$ unique soln. to the IVP $y(t) \to 0$ as $t \to +\infty$

12. $y'' + 3y' = 0 \qquad y(0) = -2, \; y'(0) = 3$

$r^2 + 3r = 0$

$r = 0 \quad r = -3$ } $y_1(t) = 1$, $y_2(t) = e^{-3t}$ } $y(t) = C_1 + C_2 e^{-3t}$

IVP: $y(0) = -2 = C_1 + C_2 \qquad C_1 = -1 \qquad y(t) = -(1 + e^{-3t})$

$y'(0) = 3 = -3C_2 \qquad C_2 = -1$

$$\lim_{t \to +\infty} y(t) = -1.$$

3.2.2. $f(t) = \cos t \qquad f'(t) = -\sin t$

$g(t) = \sin t \qquad g'(t) = \cos t$

$W(f,g)(t) = \cos^2 t + \sin^2 t = 1$

$= fg' - f'g$ independent on $\mathbb{R}$

4. $f(x) = x \qquad f'(x) = 1$

$g(x) = xe^x \qquad g'(x) = e^x(1+x) \qquad W(f,g)(x) = xe^x + x^2 e^x - xe^x$

$= x^2 e^x$

8. $(t-1)y'' - 3ty' + 4y = \sin t$ initial cond. at $t=3$
Put in standard form:
$y'' - \frac{3t}{t-1}y' + \frac{4}{t-1}y = \frac{\sin t}{t-1}$ $p(t) = -\frac{3t}{t-1}$ discont. at $t=1$
so for an initial cond at $t=3$ the soln. will exist on $(1, \infty)$.

13. $y_1(t) = t^2$ $y_2(t) = t^{-1}$ both solve $t^2 y'' - 2y = 0$ on $t>0$
Check: $y_1'(t) = 2t$ $y_1''(t) = 2$.
Subst: $t^2 y_1'' - 2y_1 = 2t^2 - 2t^2 = 0$ ✓
$y_2'(t) = -t^{-2}$ $y_2''(t) = 2t^{-3}$
Subst: $t^2 y_2'' - 2y_2 = 2t^{-1} - 2t^{-1} = 0$ ✓
The ODE is <u>linear</u>: $y = c_1 y_1 + c_2 y_2$ solves
$t^2 y'' - 2y = c_1(t^2 y_1'' - 2y_1) + c_2(t^2 y_2'' - 2y_2)$
$= 0$.

22. $y'' + 4y' + 3y = 0$. $t_0 = 1$
Find soln. y_1 & y_2 s.t. $y_1(1) = 1$ $y_1'(1) = 0$
$y_2(1) = 0$ $y_2'(1) = 1$
Note that $W(y_1, y_2)(t_0) = 1$
$r^2 + 4r + 3 = (r+3)(r+1) = 0$ Take $w_1(t) = e^{-3t}$
$w_2(t) = e^{-t}$

$y(t) = c_1 e^{-3t} + c_2 e^{-t}$
$y'(t) = -3c_1 e^{-3t} - c_2 e^{-t}$

<u>$y_1(t)$</u> $y(1) = c_1 e^{-3} + c_2 e^{-1} = 1$
$y'(1) = -3c_1 e^{-3} - c_2 e^{-1} = 0$ } $c_1 = -\frac{1}{2}e^3$ $c_2 = \frac{3}{2}e$
$y_1(t) = -\frac{1}{2}e^{3(1-t)} + \frac{3}{2}e^{1-t}$

<u>$y_2(t)$</u> $y(1) = c_1 e^{-3} + c_2 e^{-1} = 0$
$y'(1) = -3c_1 e^{-3} - c_2 e^{-1} = 1$ } $c_1 = -\frac{1}{2}e^3$ $c_2 = \frac{1}{2}e$
$y_2(t) = -\frac{1}{2}e^{3(1-t)} + \frac{1}{2}e^{1-t}$
Check these!

3.3

2. $f(\theta) = \cos 2\theta - 2\cos^2\theta = \cos^2\theta - \sin^2\theta - 2\cos^2\theta = -1$

$f'(\theta) = 0$

$g(\theta) = \cos 2\theta + 2\sin^2\theta = \cos^2\theta - \sin^2\theta + 2\sin^2\theta = 1$

$g'(\theta) = 0$

$W(f,g)(\theta) = 0$! Dependent: $f = -g$.

4. $f(x) = e^{3x}$ $\quad g(x) = e^{3(x-1)} = e^{-3}e^{3x}$ so $g = e^{-3} f$

Dependent $W(f,g) = 0$!

6. $f(t) = t$ $\quad g(t) = t^{-1}$
$f'(t) = 1$ $\quad g'(t) = -2t^{-2}$ $\Big\}$ $W(f,g)(t) = -2t^{-1} - t^{-1} = -3t^{-1}$

(f,g) indep on $(-\infty, 0)$ & $(0,\infty)$ for $t \neq 0$ never ~~zero~~

10. $W(t) = t^2 - 4$ the 2 fncs are not dependent (indep.)
since $W(t=0) = -4 \neq 0$.

16. $\cos t\, y'' + \sin t\, y' - t y = 0$

Standard form: $\quad y'' + \tan t\, y' - \dfrac{t}{\cos t} y = 0$

$p(t) = \tan t$

continuous as long as $\cos t \neq 0$ i.e. at t except $t = \dfrac{(2k+1)\pi}{2}$.

$W(t) = Ce^{-\int p(t)dt}$ $\quad$ Abel's formula.

$$\int \tan t\, dt = \int \frac{\sin t}{\cos t} dt \qquad u = \cos t \quad du = -\sin t\, dt$$

$$= -\ln|\cos t|$$

$$e^{-\int p(t)dt} = |\cos t|$$

$$W(t) = C|\cos t|.$$