MA 214

(5-1)

Problem Set 5

Homework Solutions

§ 3.6. pg 184

2. $y'' + 2y' + 5y = 3\sin 2t$

Guess: $y_p(t) = A\sin 2t + B\cos 2t$

$y_p'(t) = 2A\cos 2t - 2B\sin 2t$

$y_p''(t) = -4A\sin 2t - 4B\cos 2t$

Substitute:

$y_p'' + 2y_p' + 5y_p = (-4A + 4B + 5A)\sin 2t + (-4B + 4A + 5B) \times$ $\sin 2t = 3\sin 2t$. 2 eqns: $\begin{cases} A - 4B = 3 \\ B - 4A = 0 \end{cases}$

so $A = -1/5$ $B = -4/5$ and $y_p(t) = \frac{-1}{5}(\sin 2t + 4\cos 2t)$

Homog. ODE: $r^2 + 2r + 5 = 0$ $r_1, r_2 = -1 \pm 2i$

2 real solns: $e^{-t}\sin 2t$, $e^{-t}\cos 2t$

General Soln: $y(t) = C_1 e^{-t}\sin 2t + C_2 e^{-t}\cos 2t - \frac{1}{5}(\sin 2t + 4\cos 2t)$

6. $y'' + 2y' + y = 2e^{-t}$

Guess: $y_p(t) = Ae^{-t}$ so $y_p'(t) = -Ae^{-t}$ $y_p'' = Ae^{-t}$

Subst.: $A - 2A + A = 0 \neq 2e^{-t}$ What went wrong?

Homog ODE, $r^2 + 2r + 1 = (r+1)^2 = 0$ so e^{-t} is a soln of the homog. ODE

Solns of homog. ODE e^{-t} & te^{-t} (Double root)

Try: $y_p(t) = At^2e^{-t}$: $y_p'(t) = (2t - t^2)Ae^{-t}$

$y_p''(t) = (t^2 - 4t + 2)Ae^{-t}$

Subst.: $\left[(t^2 - 4t + 2) + 2(2t - t^2) + t^2\right]Ae^{-t} \overset{?}{=} 2e^{-t}$

$2Ae^{-t} = 2e^{-t}$ so $A = 1$ and $t^2e^{-t} = y_p(t)$

General Soln. $y(t) = C_1e^{-t} + C_2te^{-t} + t^2e^{-t}$.

14. $y'' + 4y = t^2 + 3e^t$. Homog ODE has soln. $\cos 2t$, $\sin 2t$.

For y_p. Solve $y'' + 4y = t^2$ and $y'' + 4y = 3e^t$.

First guess $At^2 + Bt + C$ so $y_p' = 2tA + B$ $y_p'' = 2A$

Subst: $2A + 4(At^2 + Bt + C) = t^2$ $B = 0$ $A = \frac{1}{4}$ $C = \frac{-1}{8}$

$y_p(t) = \frac{1}{4}t^2 - \frac{1}{8}$

Second $y''+4y=3e^t$ guess $y_p(t)=Ae^t = \frac{3}{5}e^t$.

$A+4A=3$ so $A=3/5$

General Soln: $y(t)=C_1\cos 2t + C_2\sin 2t + \left(\frac{1}{4}t^2-\frac{1}{8}\right)+\frac{3}{5}e^t$

IVP: $y(0)=0$ $\quad y'(t)=-2C_1\sin 2t + 2C_2\cos 2t + \frac{1}{2}t + \frac{3}{5}e^t$

$y'(0)=2$

$y(0)=C_1-\frac{1}{8}+\frac{3}{5}=0$

$y'(0)=2C_2+\frac{3}{5}=2$

$$\boxed{\begin{aligned}C_1&=-19/40\\ C_2&=7/10.\end{aligned}}$$

§3.7. pg 190 2. $y''-y'-2y=2e^{-t}$. Homog. ODE $r^2-r-2=0=(r-2)(r+1)$

$y_1(t)=e^{2t}$ $\quad y_2(t)=e^{-t}$ $\quad W(y_1,y_2)(t)=\begin{vmatrix} e^{2t} & e^{-t}\\ 2e^{2t} & -e^{-t}\end{vmatrix}=-3e^{t}$

$u_1'=-\frac{gy_2}{W}=-\frac{2e^{-t}\cdot e^{-t}}{-3e^t}=\frac{2}{3}e^{-3t} \Rightarrow u_1(t)=-\frac{2}{9}e^{-3t}$

$u_2'=\frac{gy_1}{W}=\frac{2e^{-t}\cdot e^{2t}}{-3e^t}=-\frac{2}{3} \Rightarrow u_2(t)=-\frac{2}{3}t$

$y_p(t)=u_1y_1+u_2y_2=-\frac{2}{9}e^{-3t}e^{2t}-\frac{2}{3}te^{-t}=\underbrace{-\frac{2}{9}e^{-t}}_{\text{soln to homog ODE}}-\frac{2}{3}te^{-t}$

$$\boxed{\text{So } y(t)=C_1e^{2t}+C_2e^{-t}-\frac{2}{3}te^{-t}}$$

13. $t^2y''-2y=3t^2-1$. Verify $y_1(t)=t^2$ & $y_2(t)=t^{-1}$ solve $t^2y''-2y=0$: $y_1'(t)=2t$, $y_1''(t)=2$ so $t^2y_1''-2y_1=2t^2-2t^2=0$

Similarly, $y_2'(t)=-t^{-2}$, $y_2''(t)=2t^{-3}$ so $t^2y_2''-y_2 2=2/t-2/t$ $=0$. Use Variation of Param. to find y_p.

$W(y_1,y_2)(t)=\begin{vmatrix} t^2 & t^{-1}\\ 2t & -t^{-2}\end{vmatrix}=-3$ $\quad u_1'=-\frac{(3t^2-1)t^{-1}}{-3}=t-\frac{1}{3t}$

$\Rightarrow u_1=\frac{1}{2}t^2-\frac{1}{3}\ln t$. $\quad u_2'=\frac{(3t^2-1)t^2}{-3}=-t^4+\frac{1}{3}t^2$

$u_2=-\frac{1}{5}t^5+\frac{1}{9}t^3$ so $y_p(t)=\frac{1}{2}t^4-\frac{1}{3}t^2\ln t-\frac{1}{5}t^4+\underbrace{\frac{1}{9}t^2}_{\text{solves homog ODE}}$

$$\boxed{y_p(t)=\frac{3}{10}t^4-\frac{1}{3}t^2\ln t}$$