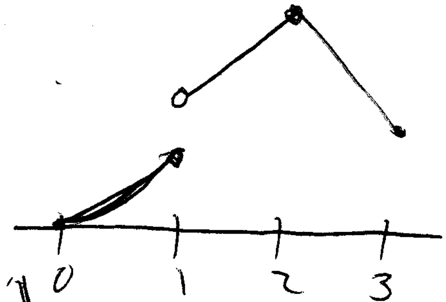
MA 214 Problems §6.1–6.2 (6-1)

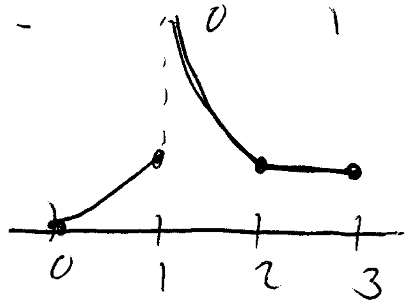
Laplace Transforms

§6.1 1. $f(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ 2+t & 1 < t \le 2 \\ 6-t & 2 < t \le 3 \end{cases}$



piecewise cont.

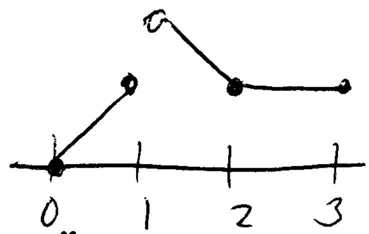
2. $f(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ (t-1)^{-1} & 1 < t \le 2 \\ 1 & 2 < t \le 3 \end{cases}$



not piecewise cont.

since $\lim\limits_{t \to 1^+} f(t) = +\infty$.

4. $f(t) = \begin{cases} t & 0 \le t \le 1 \\ 3-t & 1 < t \le 2 \\ 1 & 2 < t \le 3 \end{cases}$



piecewise cont.

5a) $(\mathcal{L}f)(s) = 1/s^2$, $s > 0$

$$\lim_{M\to\infty}\int_0^M t e^{-st}\,dt = \lim_{M\to\infty} \frac{-d}{ds}\int_0^M e^{-st}\,dt$$

for t^2 use $\displaystyle\int_0^M \frac{d^2}{ds^2} e^{-st}\,dt$

$$= \lim_{M\to\infty} \frac{-d}{ds}\frac{(e^{-sM}-1)}{(-s)} = \frac{1}{s^2}$$

6. $$\int e^{-st}\cos at = \frac{e^{-st}}{a}\sin at + \frac{s}{a}\int e^{-st}\sin at\,dt \quad (s>0)$$

$$= \frac{e^{-st}}{a}\sin at + \frac{s}{a}\left\{ -\frac{e^{-st}}{a}\cos at - \frac{s}{a}\int e^{-st}\cos at\,dt \right\}$$

So $$\int e^{-st}\cos at\,dt = \left\{\frac{e^{-st}}{a}\sin at - \frac{s}{a^2}e^{-st}\cos at\right\}\frac{a^2}{a^2+s^2}$$

Evaluate $$\lim_{M\to\infty}\int_0^M e^{-st}\cos at\,dt = \frac{s}{a^2+s^2}, \quad s>0$$

7. $$\int e^{-st}\cosh bt = \frac{1}{2}\left[\int e^{-(s-b)t}\,dt + \int e^{-(s+b)t}\,dt\right]$$

$$= \frac{e^{-(s-b)t}}{2(b-s)} - \frac{e^{-(b+s)t}}{2(b+s)}$$

$$\lim_{M\to\infty}\int_0^M e^{-st}\cosh bt\,dt = \frac{1}{2(s-b)} + \frac{1}{2(s+b)} \quad \text{if } \begin{matrix} s>b \\ s>-b \end{matrix}$$

$$= \frac{s}{(s^2-b^2)}, \quad s > |b|$$

11. $\int e^{-st}\sin bt = \frac{1}{2i}\int\left(e^{-(s-ib)t} - e^{-(s+ib)t}\right)dt$

$$= \frac{1}{2i}\left\{\frac{-e^{-(s-ib)t}}{(s-ib)} + \frac{e^{-(s+ib)t}}{s+ib}\right\}$$

Evaluate $\lim_{M\to\infty}\int_0^M e^{-st}\sin bt\,dt = \frac{b}{s^2+b^2}$.

15 $\int e^{-st}e^{at}t\,dt = \int e^{-(s-a)t}\,t\,dt \qquad \begin{matrix} u=t & dv = e^{-(s-a)t}dt \\ du=dt & v=\frac{e^{-(s-a)t}}{a-s}\end{matrix}$

$$= \frac{te^{-(s-a)t}}{a-s} - \frac{1}{a-s}\int e^{-(s-a)t}\,dt$$

$$= \frac{te^{-(s-a)t}}{a-s} - \frac{e^{-(s-a)t}}{(a-s)^2} \qquad \text{Take } \lim_{M\to\infty}\int_0^M e^{-st}(te^{at})\,dt$$

$= \frac{1}{(a-s)^2}$ provided $s>a$. Recall if $a=0$ the LT is $\frac{1}{s^2}$

so this is a shift by a.

21. $\int(t^2+1)^{-1}dt = \tan^{-1}t$ and $\int_0^M(t^2+1)^{-1}dt = \tan^{-1}M - 0$

and $\lim_{M\to\infty}\int_0^M(t^2+1)^{-1}dt = \pi/2$ converges.

§6.2. 1. $\frac{3}{s^2+4} \to \frac{3}{2}\left(\frac{2}{s^2+2^2}\right) \to \frac{3}{2}\sin 2t$

2. $\frac{4}{(s-1)^3} \qquad t^2 \xrightarrow[LT]{} \frac{2!}{s^3}$ so $e^t t^2 \to \frac{2}{(s-1)^3}$

so the ILT is $2e^t t^2$ (exponents in t shift s)

6-3

4. $F(s) = \frac{3s}{s^2-s-6} = \frac{3s}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$ or $3s = A(s+2) + B(s-3)$

$3s = (A+B)s + 2A - 3B$

$3 = A+B$, $A = \frac{3}{2}B$; $3 = \frac{5}{2}B$, $B = 6/5$, $A = 9/5$

$F(s) = \frac{9}{5}\frac{1}{s-3} + \frac{6}{5}\frac{1}{s+2}$

$f(t) = \frac{9}{5}e^{3t} + \frac{6}{5}e^{-2t}$

6. $\frac{2s-3}{s^2-4} = 2\frac{s}{s^2-4} - \frac{3}{s^2-4} = 2\cosh 2t - \frac{3}{2}\sinh 2t$

8. $\frac{8s^2-4s+12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} \Rightarrow 8s^2-4s+12 = A(s^2+4) + Bs^2 + Cs$

$8 = A+B$ $\quad B=5$

$-4 = C$

$4A = 12$ $\quad A = 3$

$= \frac{3}{s} + 5\left(\frac{s}{s^2+4}\right) - \frac{4}{2}\left(\frac{2}{s^2+2^2}\right)$

$\Rightarrow 3 + 5\cos 2t - 2\sin 2t$

12. $y'' + 3y' + 2y = 0 \quad y(0) = 1,\ y'(0) = 0$

Take LT $\quad (s^2+3s+2)(\mathcal{L}y)(s) - sy(0) - y'(0) - 3y(0) = 0$

$(s^2+3s+2)\mathcal{L}(y)(s) = s+3$.

$\mathcal{L}(y)(s) = \frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \Rightarrow s+3 = A(s+2) + B(s+1)$

$1 = A+B$, $3 = 2A+B$ $\Rightarrow -2 = -A$, $A = 2$

and $B=-1$ so $(\mathcal{L}y)(s) = \frac{2}{s+1} - \frac{1}{s+2}$.

Take ILT: $\quad y(t) = 2e^{-t} - e^{-2t}$. Check: $y(0) = 2-1 = 1$

$y'(t) = -2e^{-t} + 2e^{-2t}$ $\quad y'(0) = -2+2 = 0$.

14. $y'' - 4y' + 4y = 0$, $y(0) = 1$ $y'(0) = 1$.

$(s^2-4s+4)(\mathcal{L}y)(s) - sy(0) - y'(0) + 4y(0) = 0$

$(\mathcal{L}y)(s) = \frac{s-3}{(s-2)^2} = \frac{s-2}{(s-2)^2} - \frac{1}{(s-2)^2} = \frac{1}{s-2} - \frac{1}{(s-2)^2}$

$y(t) = e^{2t} - te^{2t}$ (use #11 pg 319)

(double root so we know e^{2t} & te^{2t} are the 2 indep.