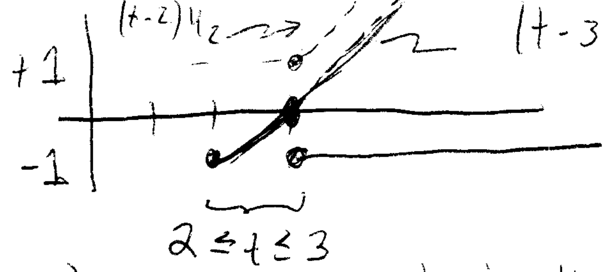
7.1 Problem Set #7

MA 214

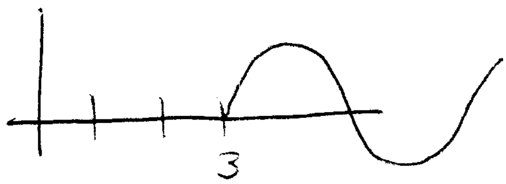
Solutions to Some Problems in §6.3-6.4.

pg. 329 2. $(t-3)u_2(t) - (t-2)u_3(t)$ so for $t > 3$:

$(t-3)u_2(t)$ $\to$ $t-3-(t-2) = -1$



4. $\sin(t-3)u_3(t)$ this starts the sine curve at $t=3$



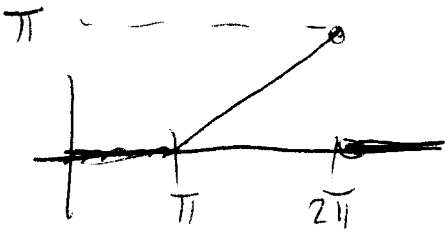
7. $f(t) = \begin{cases} 0 & 0 \le t < 2 \\ (t-2)^2 & t \ge 2 \end{cases}$ so $f(t) = u_2(t)(t-2)^2$

$(\mathcal{L}f)(s) = e^{-2s}(\mathcal{L}g)(s)$ where $g(t) = t^2$

$(\mathcal{L}g)(s) = \dfrac{2}{s^3}$ so $(\mathcal{L}f)(s) = \dfrac{2e^{-2s}}{s^3}$

(#3 pg 319)

9. $f(t) = \begin{cases} 0 & t < \pi \\ t-\pi & \pi \le t < 2\pi \\ 0 & t \ge 2\pi \end{cases}$



so f acts only on $[\pi, 2\pi)$: $f(t) = u_\pi(t)(t-\pi) - u_{2\pi}(t)(t-\pi)$

Note that for $t > 2\pi$ $f(t) = (t-\pi) - (t-\pi) = 0$.

$(\mathcal{L}f)(s) = \dfrac{e^{-s\pi}}{s^2} - \dfrac{e^{-2\pi s}}{s^2}$ since $g(t) = t$ then $(\mathcal{L}g)(s) = \dfrac{1}{s^2}$.

13. $F(s) = \dfrac{3!}{(s-2)^4}$ shift of $\dfrac{3!}{s^4} \to t^3$ so $f(t) = e^{2t}t^3$ (#11 p. 319)

14. $F(s) = \dfrac{e^{-2s}}{s^2+s-2} = \dfrac{e^{-2s}}{(s+2)(s-1)} = e^{-2s}\left(\left(\dfrac{-1}{3}\right)\dfrac{1}{s+2} + \dfrac{1}{3}\left(\dfrac{1}{s-1}\right)\right) = -\dfrac{1}{3}\dfrac{e^{-2s}}{s+2}$

$+ \dfrac{1}{3}\dfrac{e^{-2s}}{s-1} \Rightarrow f(t) = -\dfrac{1}{3}e^{-2(t-2)}u_2(t) + \dfrac{1}{3}u_2(t)e^{(t-2)}$ (#13, pg 319)

15. $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2-2s+2} = 2e^{-2s}H(s)$, $H(s) = \dfrac{s-1}{(s-1)^2+1}$

$(\mathcal{L}^{-1}H)(t) = e^t\cos t$ so $f(t) = 2u_2(t)\,e^{(t-2)}\cos(t-2)$.

16. $F(s) = \dfrac{2e^{-2s}}{s^2-4} = \dfrac{2e^{-2s}}{4}\left(\dfrac{1}{s-2} - \dfrac{1}{s+2}\right) = \dfrac{1}{2}e^{-2s}H(s)$

$(\mathcal{L}^{-1}H)(t) = e^{2t} - e^{-2t}$ so $f(t) = \dfrac{1}{2}u_2(t)\left[e^{2(t-2)} - e^{-2(t-2)}\right]$

§ 6.4. pg 337

1. $y'' + y = 1 - u_{\pi/2}(t)$ so take LT: $(s^2+1)(\mathcal{L}y)(s) - 1 = \dfrac{1}{s} - \dfrac{e^{-\pi/2\, s}}{s}$

$y(0) = 0$

$y'(0) = 1$ $\quad (\mathcal{L}y)(s) = \dfrac{1 - e^{-\pi/2\, s}}{s(s^2+1)} = H(s) - e^{-\pi/2\, s}H(s)$

$H(s) = \dfrac{1}{s(s^2+1)} = \dfrac{-s}{s^2+1} + \dfrac{1}{s}$ $\quad (\mathcal{L}^{-1}H)(t) = -\cos t + 1$

$y(t) = (1 - \cos t) - u_{\pi/2}(t)(1 - \cos(t - \pi/2))$.

3. $y'' + 4y = \sin t - u_{2\pi}(t)\sin(t - 2\pi)$

$y(0) = 0 = y'(0)$ $\quad (\mathcal{L}y)(s) = \left(\dfrac{1}{s^2+4}\right)\left(\dfrac{1}{s^2+1} - \dfrac{e^{-2\pi s}}{s^2+1}\right) = H(s) - e^{-2\pi s}H(s)$

$H(s) = \dfrac{1}{(s^2+4)(s^2+1)} = \left(-\dfrac{1}{3}\right)\left(\dfrac{1}{s^2+4} - \dfrac{1}{s^2+1}\right)$ $\quad (\mathcal{L}^{-1}H)(t) = -\dfrac{1}{6}\sin 2t + \dfrac{1}{3}\sin t$

$y(t) = -\dfrac{1}{6}\sin 2t + \dfrac{1}{3}\sin t - u_{2\pi}(t)\left(-\dfrac{1}{6}\sin 2(t-2\pi) + \dfrac{1}{3}\sin(t-2\pi)\right)$

5. $y'' + 3y' + 2y = f(t) = 1 - u_{10}(t)$ $\quad s^2 + 3s + 2 = (s+1)(s+2)$

$y(0) = 0 = y'(0)$

$(\mathcal{L}y)(s) = \dfrac{1}{s^2+3s+2}\left(\dfrac{1}{s} - \dfrac{e^{-10s}}{s}\right)$, $H(s) = \dfrac{1}{s(s+1)(s+2)} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{s+2}$

$A = \frac{1}{2}$, $A + B + C = 0$, $3A + 2B + C = 0$ so $A = \frac{1}{2}$ $B = -1$ $C = \frac{1}{2}$

$(\mathcal{L}^{-1}H)(t) = \dfrac{1}{2} - e^{-t} + \dfrac{1}{2}e^{-2t}$

$y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - u_{10}(t)\left(\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)}\right)$.