Section 6.5.

1. $y'' + 2y' + 2y = \delta(t-\pi)$ $\quad y(0)=1 \quad y'(0)=0$.

$(s^2+2s+2)(\mathcal{L}y)(s) - s = e^{-\pi s}$

$(\mathcal{L}y)(s) = \dfrac{1}{s^2+2s+2}\left(e^{-\pi s} + s\right) = \dfrac{s}{(s+1)^2+1} + e^{-\pi s}H(s)$

$s^2+2s+2 = (s+1)^2+1$

$H(s) = \dfrac{1}{(s+1)^2+1}$ $\quad (\mathcal{L}^{-1}H)(t) = e^{-t}\sin t$

$\uparrow$ shift use $\dfrac{1}{s^2+1}$

so $y(t) = e^{-t}\cos t + u_\pi(t)\, e^{-(t-\pi)}\sin(t-\pi)$

from $s/((s+1)^2+1)$

2. $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi)$ $\quad y(0) = 0 = y'(0)$

$(s^2+4)(\mathcal{L}y)(s) = e^{-\pi s} - e^{-2\pi s}$

$(\mathcal{L}y)(s) = \dfrac{e^{-\pi s}}{s^2+4} - \dfrac{e^{-2\pi s}}{s^2+4}$

$y(t) = \dfrac{1}{2}u_\pi(t)\sin 2(t-\pi) - \dfrac{1}{2}u_{2\pi}(t)\sin 2(t-2\pi)$

3. $y'' + 3y' + 2y = \delta(t-5) + u_{10}(t)$ $\quad y(0)=0 \quad y'(0)=\frac{1}{2}$

$(s^2+3s+2)(\mathcal{L}y)(s) - \frac{1}{2} = e^{-5s} + \dfrac{e^{-10s}}{s}$

$(\mathcal{L}y)(s) = \dfrac{e^{-5s}}{(s+1)(s+2)} - \dfrac{1}{2(s+1)(s+2)} + \dfrac{e^{-10s}}{s(s+1)(s+2)}$

$\dfrac{e^{-5s}}{(s+1)(s+2)} = -\dfrac{e^{-5s}}{s+2} + \dfrac{e^{-5s}}{s+1} \rightarrow -e^{-2(t-5)}u_5(t) + u_5(t)e^{-(t-5)}$

$-\dfrac{1}{2(s+1)(s+2)} = +\dfrac{1}{2}\left(\dfrac{1}{s+2} - \dfrac{1}{s+1}\right) \rightarrow \dfrac{1}{2}(e^{-2t} - e^{-t})$

$\dfrac{e^{-10s}}{s(s+1)(s+2)} = e^{-10s}\left(\dfrac{1}{2s} - \dfrac{1}{s+1} + \dfrac{1}{2(s+2)}\right) \rightarrow u_{10}(t)\left[\dfrac{1}{2} - e^{-(t-10)} + \dfrac{1}{2}e^{-2(t-10)}\right]$

$y(t)$: add use the terms