NAME: Solutions

1. (5 points). Find a solution of the initial value problem:

\[ M(x, y) = xy^2 \]
\[ N(x, y) = 4 + yx^2 \]

\[ xy^2 + (4 + yx^2)y'(x) = 0, \quad y(0) = 1. \]

**Solution for \( \psi(x, y) \):**

\[ \frac{2\psi}{dx} = M, \quad \frac{2\psi}{dy} = N \]

Yes. \( \Rightarrow \psi = \frac{1}{2} x^2 y^2 + h(y) \)

\[ y = 4y + \frac{1}{2} y^2 x^2 + g(x) \]

We need \( h(y) = 4y \) and \( g(x) = 0 \).

Solve: \( y = C \)

\( y(0) = 1 \) \( \Rightarrow C = 4 \) so

\[ y = 4y + \frac{1}{2} x^2 y^2 + x y^2 \]

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2. (5 points). A tank holds 100 liters. A solution pours into the tank at a rate of 2 liters/sec and has a concentration of pollutant given by 0.01 grams/liter. The solution mixes uniformly in the tank and flows out at the same rate. If the tank initially contains pure water, find the amount of pollutant in the tank as a function of \( t \) in seconds. How much pollutant is in the tank after 50 sec?

\[ \left\{ \begin{array}{l}
\frac{dQ}{dt} = \frac{2L}{5} \cdot 10^{-2} \frac{g}{L} - \frac{2L}{5} \cdot \frac{Q(t)}{100} \\
Q(t) = \text{amount of pollutant in the tank at time } t \text{ in grams.} \\
Q(t=0) = 0
\end{array} \right. \]

Solve:

\[ \frac{dQ}{dt} + \frac{1}{50} Q(t) = \frac{1}{50} \]

\( p = \frac{1}{50}, \quad g = \frac{1}{50} \) integrating factor

\[ M(t) = e^{t/50} \]

\[ Q(t) e^{t/50} = \frac{1}{50} \int e^{t/50} dt = e^{t/50} + C \]

\[ Q(t) = 1 + Ce^{-t/50} \]

\( Q(0) = 0 \) \( \Rightarrow C = -1 \)

\[ Q(t) = 1 - e^{-t/50} \]

\( Q(50) = 1 - e^{-1} \)

Check: \( Q'(t) = \frac{1}{50} e^{-t/50} \)

\[ Q' + \frac{1}{50} Q = \frac{1}{50} e^{-t/50} + \frac{1}{50} - \frac{1}{50} e^{-50} = 0 \]