1. (15 points). Find the solution to the ODE with the given initial condition:

\[(y + x^2 y) y'(x) = 2x, \quad y(2) = 0.\]

Separate:

\[
\frac{y (1+x^2)}{dx} = 2x
\]

\[
\int y \, dy = \int \frac{2x}{1+x^2} \, dx \quad u = 1+x^2 \quad du = 2x \, dx
\]

\[
\frac{1}{2} y^2 = \ln (1+x^2) + C
\]

Initial condition:

\[
\frac{1}{2} y(2)^2 = 0 = \ln (1+4) + C \quad \Rightarrow \quad C = -\ln 5
\]

\[
y^2 = 2 \ln (1+x^2) - 2 \ln 5
\]

We can't take the square root since we don't know if \( y/x \) is positive or negative.
2. (15 points). A tank initially contains 100 liters of pure water. A stream of polluted water with a concentration of $\gamma = 5$ grams per liter of mercury is poured into the tank at a rate of one liter per minute. It flows out at the same rate. How much mercury is in the tank after 100 minutes and 2 minutes?

\[ Q = \text{amount of mercury in tank} \]

\[ \frac{dQ}{dt} = [\text{flow in}] - [\text{flow out}] \]

\[ = 5 \frac{g}{l} \cdot 1 \frac{l}{min} - \frac{(Q(t))}{100} \frac{g}{l} \cdot 1 \frac{l}{min} \]

\[ Q' = 5 - \frac{Q}{100} \]

\[ Q' + \frac{1}{100} Q = 5 \]  

Integrating factor:

\[ M = e^{\frac{t}{100}} \]

\[ (MQ)' = MQ = 5e^{\frac{t}{100}} \]

\[ MQ = 500e^{\frac{t}{100}} + C \]

\[ Q(t) = 500 + Ce^{-\frac{t}{100}} \]

\[ C = -500 \text{ since at } t = 0 \quad Q(0) = 0 \]

\[ Q(t) = 500(1 - e^{-\frac{t}{100}}) \]

\[ Q(100 \text{ min}) = 250 \text{ gr} \]
3. (15 points). Find the unique solution to the initial value problem:

\[ ty' + 2y = 4t^2, \quad y(1) = 2. \]

For what interval of time about \( t = 1 \) is the solution valid?

\[ y' + \frac{2}{t} y = 4t \]

\[ p(t) = \frac{2}{t}, \quad \int p(t) dt = 2 \ln t \quad M = e^{2 \ln t} = t^2 \]

\[ (\mu y)' = \mu q = 4t^3 \]

\[ \mu y = t^4 + C \quad \text{so} \quad y(t) = t^2 + \frac{C}{t^2} \quad \text{general soln ok for } t \neq 0. \]

If we have \( y(1) = 2 \), then

\[ 2 = 1 + C \quad \text{so} \quad C = 1 \]

\[ y(t) = t^2 + \frac{1}{t^2} \quad \text{and the soln is good on } (0, \infty). \]
4. (20 points). The population \( P(t) \) of cave bats changes according to the logistic equation:

\[
P'(t) = kP(t)(R - P(t)) \quad (r > 0, \ k > 0).
\]

If the initial population is \( P(t = 0) = P_0 < R \), find the population at time \( t \).
What is the stable population obtained when \( t \to \infty \)?

Separate:
\[
\frac{dP}{P(R-P)} = k \, dt
\]

Partial fraction:
\[
\frac{1}{P(R-P)} = \frac{A}{P} + \frac{B}{R-P}
\]

\[1 = A(R-P) + BP\]
\[-(A+B)P + AR = 0\]
\[A = B \quad \text{and} \quad A = \frac{1}{R}\]

\[
\int \frac{dP}{P(R-P)} = \int \frac{1}{R} \ln P - \frac{1}{R} \ln (R-P)
\]
\[= \frac{1}{R} \ln \left( \frac{P}{R-P} \right) \quad (0 < P < R)
\]

\[\ln \left( \frac{P}{R-P} \right) = Rkt + C \Rightarrow \frac{P(t)}{R - P(t)} = Ce^{Rkt} \text{ general solution}
\]

\[t=0 \quad \frac{P_0}{R-P_0} = C
\]

\[
\frac{P}{R-P} = \left( \frac{P_0}{R-P_0} \right) e^{Rkt}
\]

Solve for \( P(t) \):
\[
P(t) = \frac{R \frac{P_0}{R-P_0} e^{Rkt}}{1 + \frac{P_0}{R-P_0} e^{Rkt}}
\]

\[\text{Take } t \to \infty \quad \ln \left( \frac{P(t)}{t \to \infty} \right) = R \quad \text{equilibrium population} \]
5. (15 points). Find the most general solution to the ODE:

\[ e^x \sin y + (2 + e^x \cos y)y'(x) = 0. \]

Exact? \[ M = e^x \sin y \quad \frac{\partial M}{\partial y} = e^x \cos y \quad \text{equal so} \]

\[ N = e^x \cos y + 2 \quad \frac{\partial N}{\partial x} = e^x \cos y \quad \text{EXACT} \]

Find \( \psi \) so \[ \frac{\partial \psi}{\partial x} = M \quad \psi = \int M \, dx = e^x \sin y + g(y) \]

\[ \frac{\partial \psi}{\partial y} = N \quad \psi = \int N \, dy = 2y + e^x \sin y + h(x) \]

so \( g(y) = 2y \) \[ h(x) = 0 \quad \text{so} \quad 2y + e^x \sin y = \psi(x,y) \]

General Soln:

\[ C = 2y + e^x \sin y \]

Check: \[ 0 = 2y' + e^x \sin y + e^x \cos y \quad y' \]

\[ 0 = (2 + e^x \cos y)y' + e^x \sin y \]
6. (20 points). Consider the second-order ODE:

\[ y'' - 3y' + 2y = 0. \]

a. Find a pair of independent solutions (be sure to check that they are independent).

b. Write the most general solution to the ODE.

c. Find the unique solution satisfying the initial condition:

\[ y(0) = 1, \quad y'(0) = 1. \]

a) \[ r^2 - 3r + 2 = 0 \]

\[ (r - 2)(r - 1) = 0 \quad \therefore r_1 = 2, \quad r_2 = 1 \]

\[ \therefore y_1(t) = e^{2t}, \quad y_2(t) = e^t \]

\[ W(y_1, y_2)(t) = \begin{vmatrix} e^{2t} & e^t \\ 2e^{2t} & e^t \end{vmatrix} = -e^{3t} \neq 0 \text{ so independent on } \mathbb{R} \]

b) \[ y(t) = C_1 e^{2t} + C_2 e^t \]

c) \[ y'(t) = 2C_1 e^{2t} + C_2 e^t \]

\[ \begin{align*}
 y(0) &= C_1 + C_2 = 1 \\
y'(0) &= 2C_1 + C_2 = 1 \\

\therefore C_1 &= 0, \quad C_2 = 1
\end{align*} \]

\[ y(t) = e^t \text{ (check!)} \]

\[ y(0) = 1, \quad y'(0) = 1 \]

\[ e^t - 3e^t + 2e^t = 0 \quad \checkmark \]