MA214-003 Spring 2009
Test #1 - 100 Points
2 March 2009

INSTRUCTIONS: PLEASE WORK ALL THE PROBLEMS BELOW. PLEASE WRITE YOUR NAME ON EACH PAGE. NO CALCULATORS, BOOKS, PAPERS, OR NOTES ARE ALLOWED.

NAME: Solutions

1. (15 points). Find the solution to the ODE with the given initial condition:

$$(y + x^2 y)y'(x) = 2x, \quad y(2) = 0.$$

Separate $\quad y(1+x^2)\frac{dy}{dx} = 2x$

$$\int y\,dy = \int \frac{2x}{1+x^2}\,dx \qquad u = 1+x^2, \quad du = 2x\,dx$$

$$\boxed{\frac{1}{2}y^2 = \ln(1+x^2) + C}$$

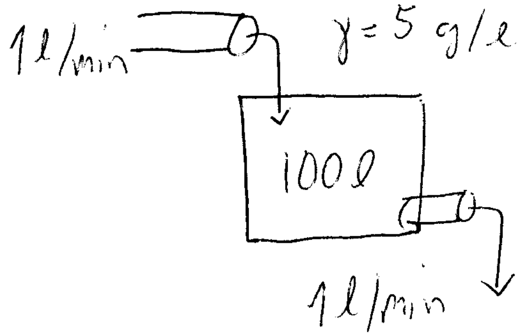
Initial Condition:

$$\frac{1}{2}y(2)^2 = 0 = \ln(1+4) + C \Rightarrow C = -\ln 5$$

$$\boxed{y^2 = 2\ln(1+x^2) - 2\ln 5}$$

We can't take the square root since we don't know if $y(x)$ is positive or negative.

2. (15 points). A tank initially contains 100 liters of pure water. A stream of polluted water with a concentration of $\gamma = 5$ grams per liter of mercury is poured into the tank at a rate of one liter per minute. It flows out at the same rate. How much mercury is in the tank after ~~10~~ minutes? after $100 \ln 2$ minutes



$Q \equiv$ amount of mercury in tank

$$\frac{dQ}{dt} = [\text{flow in}] - [\text{flow out}]$$

$$= 5\,\frac{g}{l} \cdot 1\,\frac{l}{min} - \frac{Q(t)}{100}\,\frac{g}{l} \cdot 1\,\frac{l}{min}$$

concentration

$$Q' = 5 - \frac{Q}{100}$$

$$\boxed{Q' + \frac{1}{100} Q = 5}$$

$p(t) = \frac{1}{100}$

$q(t) = 5$

Integrating factor

$$\int p(t)\,dt = \frac{t}{100}$$

$$\mu = e^{t/100}$$

$$(\mu Q)' = \mu q = 5e^{t/100}$$

$$\mu Q = 500 e^{t/100} + C$$

$$Q(t) = 500 + Ce^{-t/100}$$

$C = -500$ since at $t = 0$ $Q(0) = 0$.

$$Q(t) = 500(1 - e^{-t/100})$$

$$\boxed{Q(100 \ln 2) = 250 \text{ gr}}$$

3. (15 points). Find the unique solution to the initial value problem:

$$ty' + 2y = 4t^2, \quad y(1) = 2.$$

For what interval of time about $t = 1$ is the solution valid?

$$y' + \frac{2}{t} y = 4t$$

$p(t) = \frac{2}{t}$ $\int p(t)\,dt = 2\ln t$ $\mu = e^{2\ln t} = t^2$

$(\mu y)' = \mu q = 4t^3$

$\mu y = t^4 + C$ so $\boxed{y(t) = t^2 + \frac{C}{t^2}}$ general soln OK for $t \neq 0$.

If we have $y(1) = 2$, then

$2 = 1 + C$ so $C = 1$

$\boxed{y(t) = t^2 + \frac{1}{t^2}}$ and the soln is good on $(0, \infty)$.

4. (20 points). The population $P(t)$ of cave bats changes according to the logistic equation:

$$P'(t) = kP(t)(R - P(t)), \quad r > 0, \quad k > 0.$$

If the initial population is $P(t=0) = P_0 < R$, find the population at time t. What is the stable population obtained when $t \to \infty$?

Separate: $\dfrac{dP}{P(R-P)} = k\,dt$

Partial fraction: $\dfrac{1}{P(R-P)} = \dfrac{A}{P} + \dfrac{B}{R-P}$

$$1 = A(R-P) + BP$$
$$= (-A+B)P + AR \qquad A = B \text{ and } A = 1/R$$

$$\int \frac{dP}{P(R-P)} = \frac{1}{R}\ln P - \frac{1}{R}\ln(R-P) \qquad (0 < P < R)$$
$$= \frac{1}{R}\ln\left(\frac{P}{R-P}\right)$$

so $\ln\left(\dfrac{P}{R-P}\right) = Rkt + C \Rightarrow \boxed{\dfrac{P(t)}{R-P(t)} = Ce^{Rkt}}$ general solution

$\underline{t=0}$ $\quad \dfrac{P_0}{R-P_0} = C \qquad \dfrac{P}{R-P} = \left(\dfrac{P_0}{R-P_0}\right)e^{Rkt}$

Solve for $P(t)$: $\boxed{P(t) = \dfrac{\frac{RP_0}{R-P_0}e^{Rkt}}{1 + \frac{P_0}{R-P_0}e^{Rkt}}}$

Take $t \to +\infty$ $\boxed{\lim_{t \to +\infty} P(t) = R}$ equilibrium population.

5. (15 points). Find the most general solution to the ODE:

$$e^x \sin y + (2 + e^x \cos y)y'(x) = 0.$$

Exact? $M = e^x \sin y \quad \frac{\partial M}{\partial y} = e^x \cos y$

$N = e^x \cos y + 2 \quad \frac{\partial N}{\partial x} = e^x \cos y$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ equal so EXACT

Find ψ so $\frac{\partial \psi}{\partial x} = M \quad \psi = \int M\,dx = e^x \sin y + g(y)$

$\frac{\partial \psi}{\partial y} = N \quad \psi = \int N\,dy = 2y + e^x \sin y + h(x)$

so $\left.\begin{matrix} g(y) = 2y \\ h(x) = 0. \end{matrix}\right\}$ so $2y + e^x \sin y = \psi(x,y)$

General Soln:

$$\boxed{C = 2y + e^x \sin y}$$

Check: $0 = 2y' + e^x \sin y + e^x \cos y\, y'$

$0 = (2 + e^x \cos y)y' + e^x \sin y.$

6. (20 points). Consider the second-order ODE:

$$y'' - 3y' + 2y = 0.$$

a. Find a pair of independent solutions (be sure to check that they are independent).

b. Write the most general solution to the ODE.

c. Find the unique solution satisfying the initial condition:

$$y(0) = 1, \quad y'(0) = 1.$$

a) $r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0 \quad r_1 = 2 \quad r_2 = 1 \quad y_1(t) = e^{2t}, \quad y_2(t) = e^t$

$W(y_1, y_2)(t) = \begin{vmatrix} e^{2t} & e^t \\ 2e^{2t} & e^t \end{vmatrix} = -e^{3t} \neq 0$ so independent on $\mathbb{R}$

b) $y(t) = C_1 e^{2t} + C_2 e^t$

c) $y'(t) = 2C_1 e^{2t} + C_2 e^t$

$y(0) = C_1 + C_2 = 1$

$y'(0) = 2C_1 + C_2 = 1$

$C_1 = 0 \quad C_2 = 1$

$y(t) = e^t$ Check $y(0) = 1 \quad y'(0) = 1$

$e^t - 3e^t + 2e^t = 0$. ✓