MA214-003 Fall 2008
Test #1 - 100 Points
10 October 2008

INSTRUCTIONS: PLEASE WORK ALL THE PROBLEMS BELOW. PLEASE WRITE YOUR NAME ON EACH PAGE. NO CALCULATORS, BOOKS, PAPERS, OR NOTES ARE ALLOWED.

NAME:

1. (15 points). Find the general solution to the ODE:

$$xy^2 + y\frac{dy}{dx} = 0.$$

Separate: $xy^2 = -\frac{dy}{dx} y$

$$\int x\,dx = -\int \frac{1}{y}\,dy$$

$$\tfrac{1}{2}x^2 + C = -\ln|y|$$

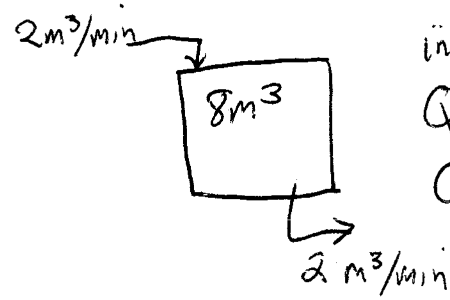
$$\boxed{C_0 e^{-\frac{1}{2}x^2} = y(x)}$$

Check: $y'(x) = -x\,y(x)$

so $xy + y' = 0$

$xy^2 + y y' = 0$ ✓

2. (20 points). A room contains 8 m^3 (cubic meters) of gas with an initial concentration of 5 g/m^3 of poisonous gas. Air enters the room at a rate of 2 m^3/min and contains a concentration of $5e^{-t/4}$ g/m^3 of poison. Poisoned air flows out of the room at the same rate. Find the quantity $Q(t)$ (in grams) of poison in the room at any time $t > 0$.



initial concentration $5 g/m^3$

$Q(t)$ amount at time $t \geq 0$

$Q(0) = 8m^3 \cdot 5\frac{g}{m^3} = 40g$

$$\frac{dQ}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= 2\frac{m^3}{min} \cdot 5\frac{g}{m^3} e^{-t/4} - 2\frac{m^3}{min}\,\frac{Q(t)}{8m^3}\,g$$

$$= 10e^{-t/4} - \frac{1}{4}Q(t) \text{ in } g/min.$$

$$\begin{cases} \frac{dQ}{dt} + \frac{1}{4}Q = 10e^{-t/4} \ g/min \\ Q(0) = 40g \end{cases}$$

$\mu(t) = e^{t/4}$ $\quad (\mu Q)' = \mu\, 10e^{-t/4} = 10$

$$e^{t/4}Q = 10t + C$$

$$Q(t) = e^{-t/4}(10t + C)$$

$$Q(0) = C = 40$$

$$\boxed{Q(t) = e^{-t/4}(10t + 40)\ g}$$

3. (15 points). Find the unique solution to the initial value problem:

$$ty' + y = e^t, \quad y(1) = 0.$$

For what interval of time about $t = 1$ is the solution valid?

$y' + \frac{1}{t}y = \frac{1}{t}e^t \qquad p(t) = \frac{1}{t} \qquad \mu(t) = e^{\int p(t)dt} = t$

$(ty)' = e^t$

$ty = e^t + C$

$y(t) = \frac{1}{t}(e^t + C)$ general soln.

$y(1) = 0 = e + C \qquad C = -e$

$$\boxed{\begin{cases} y(t) = \frac{1}{t}(e^t - e) \\ t > 0 \end{cases}}$$

Check $y' = -\frac{1}{t^2}(e^t - e) + \frac{e^t}{t}$

$y' + \frac{1}{t}y = \frac{e^t}{t}$ ✓

4. (20 points). Find two independent solutions of the ODE:

$$y'' - y' - 2y = 0.$$

Verify that the solutions are independent. Find the unique solution with initial values: $y(0) = 1$ and $y'(0) = 0$.

$r^2 - r - 2 = (r-2)(r+1) = 0$

$y_1(t) = e^{2t} \qquad y_2(t) = e^{-t}$

$W(y_1, y_2)(t) = e^{2t}(-e^{-t}) - (2e^{2t})(e^{-t})$.

$= -3e^t \neq 0$ never vanishes

so the 2 solns are independent.

General soln. $y(t) = C_1 e^{2t} + C_2 e^{-t}$

$y'(t) = 2C_1 e^{2t} - C_2 e^{-t}$

$y(0) = C_1 + C_2 = 1 \qquad C_1 = 1/3 \ \& \ C_2 = 2/3$

$y'(0) = 2C_1 - C_2 = 0 \qquad C_2 = 2C_1$

So $\boxed{y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}}$

5. (15 points). Find the unique solution to the initial value problem:

$$\left(\frac{y}{x} + 2x\right) + \ln x \frac{dy}{dx} = 0, \quad x > 0,$$

and $y(2) = 0$.

$M(x,y) = \frac{y}{x} + 2x \qquad N(x,y) = \ln x$

$\frac{\partial M}{\partial y} = \frac{1}{x} \qquad \frac{\partial N}{\partial x} = \frac{1}{x}$ so Exact

Find $\psi(x,y)$ so $\frac{\partial \psi}{\partial x} = M \Rightarrow \psi = y \ln x + x^2 + g(y)$

$\frac{\partial \psi}{\partial y} = N \Rightarrow \psi = y \ln x + h(x)$

$\Rightarrow h(x) = x^2$ & $g(y) = 0$

$\psi(x,y) = y \ln x + x^2 = C$

$$\boxed{y(x) = \frac{C - x^2}{\ln x}, \quad x > 0}$$

Initial Cond: $y(2) = 0 = \frac{C-4}{\ln 2}$

$C = 4$

$$y(x) = \frac{4 - x^2}{\ln x}, \quad x > 0$$

6. (15 points). Find the general solution to the ODE:

$$y'' + 2y' + 3y = 0.$$

Use only real functions for the general solution.

$r^2 + 2r + 3 = 0$

$r_1, r_2 = \frac{-2 \pm [4 - 12]^{\frac{1}{2}}}{2} = -1 \pm \sqrt{2}\, i$

$\tilde{y}_1(t) = e^{-t} e^{\sqrt{2} i t} = e^{-t}(\cos\sqrt{2}t + i \sin\sqrt{2}t)$

$\tilde{y}_2(t) = e^{-t} e^{-\sqrt{2} i t} = e^{-t}(\cos\sqrt{2}t + i \sin\sqrt{2}t)$

2 real soln $\mathrm{Re}\, \tilde{y}_1$ and $\mathrm{Im}\, \tilde{y}_1$:

$y_1(t) = e^{-t} \cos\sqrt{2}t$

$y_2(t) = e^{-t} \sin\sqrt{2}t$

so the general soln is

$$\boxed{y(t) = C_1 e^{-t} \cos\sqrt{2}t + C_2 e^{-t} \sin\sqrt{2}t}$$