1. (15 points). Find the general solution to the ODE:

\[ xy^2 + \frac{dy}{dx} = 0. \]

Separate:

\[ xy^2 = - \int \frac{1}{y} \, dy \]

\[ \frac{1}{2} x^2 + C = - \ln |y| \]

Check:

\[ y'(x) = - xy(x) \]

So \( xy + y' = 0 \)

\[ xy^2 + y y' = 0 \]

\[ \boxed{c e^{-\frac{1}{2} x^2} = y(x)} \]

2. (20 points). A room contains 8 m³ (cubic meters) of gas with an initial concentration of 5 g/m³ of poisonous gas. Air enters the room at a rate of 2 m³/min and contains a concentration of 5e⁻ᵗ/₄ g/m³ of poison. Poisoned air flows out of the room at the same rate. Find the quantity \( Q(t) \) (in grams) of poison in the room at any time \( t \geq 0 \).

Initial concentration: \( 5 \text{ g/m}^3 \)

\[ Q(0) = 8 \text{ m}^3 \cdot \frac{5 \text{ g}}{\text{m}^3} = 40 \text{ g} \]

\[ \frac{dQ}{dt} = \text{(rate in)} - \text{(rate out)} \]

\[ = \frac{2}{\text{min}} \cdot \frac{5 \text{ g}}{\text{m}^3} \cdot \frac{e^{-t/4}}{\text{min}} - 2 \frac{\text{m}^3}{\text{min}} \cdot \frac{Q(t)}{8 \text{ m}^3} \]

\[ = 10e^{-t/4} - \frac{1}{4} Q(t) \quad \text{g/min} \]

\[ \begin{align*}
\left\{ \frac{dQ}{dt} + \frac{1}{4} Q &= 10e^{-t/4} \quad \text{g/min} \\
Q(0) &= 40 \text{ g}
\end{align*} \]

\[ M(t) = e^{t/4} \quad (MQ)' = \mu e^{-t/4} = 10 \]

\[ e^{t/4} Q = 10t + C \]

\[ Q(t) = e^{-t/4}(10t + C) \]

\[ Q(0) = C = 40 \]

\[ Q(t) = e^{-t/4}(10t + 40) \quad \text{g} \]
3. (15 points). Find the unique solution to the initial value problem:
\[ ty' + y = e^t, \quad y(1) = 0. \]
For what interval of time about \( t = 1 \) is the solution valid?

\[ y' + \frac{1}{t} y = \frac{1}{t} e^t \quad (\lambda_1) = \frac{1}{t} \quad \mu(1) = e^{\int \frac{1}{t} dt} = e^t \]
\[ (ty)' = e^t \]
\[ ty = e^t + C \]
\[ y(t) = \frac{1}{t} (e^t + C) \quad \text{general soln}. \]
\[ y(1) = 0 = e + C \quad C = -e \]
\[ \begin{cases} 
  y(t) = \frac{1}{t} (e^t - e) \\
  t > 0 
\end{cases} \]

\[ y' + \frac{1}{t} y = \frac{e^t}{t} \quad \checkmark \]

4. (20 points). Find two independent solutions of the ODE:
\[ y'' - y' - 2y = 0. \]
Verify that the solutions are independent. Find the unique solution with initial values: \( y(0) = 1 \) and \( y'(0) = 0. \)

\[ r^2 - r - 2 = (r - 2)(r + 1) = 0 \]
\[ y_1(t) = e^{2t} \quad y_2(t) = e^{-t} \]
\[ W(y_1, y_2)(t) = e^{2t} (e^{-t}) - (2e^{2t})(e^{-t}) = -3e^t \neq 0 \quad \text{never vanishes} \]

So the 2 solutions are independent.

General soln. \[ y(t) = C_1 e^{2t} + C_2 e^{-t} \]
\[ y'(t) = 2C_1 e^{2t} - C_2 e^{-t} \]
\[ y(0) = C_1 + C_2 = 1 \quad C_1 = \frac{1}{3} \quad C_2 = \frac{2}{3} \]
\[ y'(0) = 2C_1 - C_2 = 0 \quad C_2 = 2C_1 \]
So \[ y(t) = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t} \]
5. (15 points). Find the unique solution to the initial value problem:

\[
(\frac{y}{x} + 2x) + \ln x \frac{dy}{dx} = 0, \quad x > 0,
\]

and \(g(2) = 0\).

\[
M(x,y) = \frac{y}{x} + 2x \quad N(x,y) = \ln x
\]

Also \(\frac{\partial M}{\partial y} = \frac{1}{x} \quad \frac{\partial N}{\partial x} = \frac{1}{x} \quad \text{so exact}
\]

Find \(\Psi(x,y)\) so \(\frac{\partial \Psi}{\partial x} = M \Rightarrow \Psi = y \ln x + x^2 + g(y)\)

\[\frac{\partial \Psi}{\partial y} = N \Rightarrow \Psi = y \ln x + h(x)\]

\[y(x) = C - \frac{x^2}{\ln x}, x > 0\]

(Initial condition: \(y(2) = 0 \Rightarrow \frac{C - 4}{\ln 2} = 0\))

\[C = 4\]

\[y(x) = \frac{4 - x^2}{\ln x}, x > 0\]

6. (15 points). Find the general solution to the ODE:

\[y'' + 2y' + 3y = 0\]

Use only real functions for the general solution.

\[r_1, r_2 = -2 \pm \sqrt{4 - 12} = -1 \pm \sqrt{2}i\]

\[y_1(t) = e^{-t}e^{-\frac{\sqrt{2}it}{2}} = e^{-t}(\cos \sqrt{2}t + i \sin \sqrt{2}t)\]

\[y_2(t) = e^{-t}e^{-\frac{\sqrt{2}it}{2}} = e^{-t}(\cos \sqrt{2}t + i \sin \sqrt{2}t)\]

2 real solns, Re \(y_i\) and \(\ln y_i\):

\[y_1(t) = e^{-t} \cos \sqrt{2}t\]

\[y_2(t) = e^{-t} \sin \sqrt{2}t\]

So the general solution is

\[y(t) = C_1 e^{-t} \cos \sqrt{2}t + C_2 e^{-t} \sin \sqrt{2}t\]