MA214 Spring 2009
Test #2 - 100 points
10 April 2009

NAME: Solutions

INSTRUCTIONS: PLEASE WORK ALL FOUR PROBLEMS BELOW. PLEASE WRITE YOUR NAME ON EACH PAGE. NO CALCULATORS, BOOKS, PAPERS, OR NOTES ARE ALLOWED.
Each problem is worth 25 points.

1. Find the unique solution to the initial value problem:

$$y''(t) + 4y'(t) + 4y(t) = 0, \quad y(0) = 1, y'(0) = 0.$$

$r^2 + 4r + 4 = 0 = (r+2)^2$ one root: $r = -2$

$y_1(t) = e^{-2t}$ $y_2(t) = te^{-2t}$

2 independent solns.

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t}(1-2t) \end{vmatrix} = e^{-4t} \neq 0.$$

General Soln:

$$\boxed{y(t) = C_1 e^{-2t} + C_2 t e^{-2t}}$$

$$y'(t) = -2C_1 e^{-2t} + C_2 e^{-2t} - 2C_2 t e^{-2t}$$

$y(0) = C_1 = 1$

$y'(0) = -2 + C_2 = 0$ $C_2 = 2$

Unique Soln:

$$\boxed{y(t) = e^{-2t} + 2te^{-2t}}$$

NAME: ______________________________

2. Find the most general solution to the ODE:

$$y''(t) + 2y'(t) + y(t) = \frac{e^{-t}}{t}, \quad t > 0.$$

homog. ODE: $r^2 + 2r + 1 = 0 = (r+1)^2$

2 indep solns: $y_1(t) = e^{-t}$ $\quad y_2(t) = te^{-t}$

$y_{homog}(t) = c_1 e^{-t} + c_2 t e^{-t}$

Variation of parameters:

$c_1'(t) = -\frac{y_2(t) g(t)}{W}$ $\qquad c_2'(t) = \frac{y_1(t) g(t)}{W}$.

$W(y_1, y_2)(t) = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t}(1-t) \end{vmatrix} = e^{-2t} \neq 0$.

$c_1'(t) = -\frac{te^{-t}}{e^{-2t}} \cdot \frac{e^{-t}}{t} = -1$ $\qquad c_1(t) = -t$

$c_2'(t) = \frac{e^{-t}}{e^{-2t}} \cdot \frac{e^{-t}}{t} = \frac{1}{t}$ $\qquad c_2(t) = \ln t$

$y_p(t) = c_1(t) y_1(t) + c_2(t) y_2(t) = \underbrace{-te^{-t}}_{\text{soln. to homog. ODE}} + (t \ln t) e^{-t}$

General Soln:

$$\boxed{y(t) = c_1 e^{-t} + c_2 t e^{-t} + (t \ln t) e^{-t}}$$

NAME: Solutions

3. Find the unique solution to the driven harmonic oscillator ODE with initial conditions at rest:

$$y''(t) + 4y(t) = 10e^{-t}, \quad y(0) = 0, \quad y'(0) = 0.$$

homog ODE: $r^2 + 4 = 0$ $\quad r = \pm 2i$ $\quad W = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2.$

$y_1(t) = \cos 2t \quad y_2(t) = \sin 2t$

particular soln:

Guess: $y_p(t) = Ae^{-t} \quad y_p'(t) = -Ae^{-t} \quad y_p''(t) = Ae^{-t}$

Substitute: $y_p'' + 4y_p = (A + 4A)e^{-t} = 10e^{-t}$

$A = 2$

$y_p(t) = 2e^{-t}$

General Soln: $y(t) = C_1 \cos 2t + C_2 \sin 2t + 2e^{-t}$

$y(0) = C_1 + 2 = 0 \quad C_1 = -2$

$y'(t) = -2C_1 \sin 2t + 2C_2 \cos 2t - 2e^{-t}$

$y'(0) = 2C_2 - 2 = 0$

$C_2 = 1$

$$\boxed{y(t) = -2\cos 2t + \sin 2t + 2e^{-t}}$$

Variation of Parameters $\quad g(t) = 10e^{-t}$

$c_1' = -\frac{y_2 g}{W} = -5e^{-t}\sin 2t \quad c_1(t) = -e^{-t}(-\sin 2t - 2\cos 2t)$

$c_2' = \frac{y_1 g}{W} = 5e^{-t}\cos 2t \quad c_2(t) = e^{-t}(-\cos 2t + 2\sin 2t)$

$y_p(t) = c_1 y_1 + c_2 y_2 = +e^{-t}\left[(\cos 2t \sin 2t + 2\cos^2 2t) + (-\cos 2t \sin 2t + 2\sin^2 2t)\right]$

$= 2e^{-t}$

NAME: ______________________________

4. Find the unique **real** solution to the ODE with initial conditions:

$$y''(t) - 2y'(t) + 5y(t) = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 1.$$

homog. ODE: $r^2 - 2r + 5 = 0$

roots $\dfrac{+2 \pm [4-20]^{\frac{1}{2}}}{2} = 1 \pm 2i$

2 independent real solns:

$y_1(t) = e^t \cos 2t \qquad y_2(t) = e^t \sin 2t$

(use $e^{(1\pm 2i)t} = e^t e^{\pm 2it} = e^t(\cos 2t \pm i \sin 2t)$)

General Soln:

$$\boxed{y(t) = C_1 e^t \cos 2t + C_2 e^t \sin 2t}$$

$y'(t) = C_1 e^t \cos 2t - 2C_1 e^t \sin 2t + C_2 e^t \sin 2t$
$\quad + 2C_2 e^t \cos 2t$

$y(\frac{\pi}{2}) = -C_1 e^{\pi/2} = 0 \Rightarrow C_1 = 0$

$y'(\frac{\pi}{2}) = -2C_2 e^{\pi/2} = 1 \qquad C_2 = -\frac{1}{2} e^{-\pi/2}$

$$\boxed{y(t) = -\frac{1}{2} e^{(t-\pi/2)} \sin 2t}$$